On Critical Point for Two Dimensional Holomorphics Systems

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Abstract

Let $f : M \to M$ be a biholomorphisms on two–dimensional a complex manifold , and let $X \subseteq M$ be a compact f–invariant set such that $f|X$ is asymptotically dissipative and without sinks periodic points. We introduce a solely dynamical obstruction to dominated splitting, namely critical point. Critical point is a dynamical object and capture many of the dynamical properties of their one–dimensional counterpart.

Contents

1   Introduction

Interested in a global understanding of holomorphic dynamics in higher dimensions, the polynomial automorphisms of $\mathbb{C}^2$ known as generalized Hénon maps (or complex Hénon maps), appear in a natural way in this study.

These maps were a subject of serious study with foundational work carried out by J. Hubbard [11], J. Hubbard and R. Oberste–Vorth [13, 14], E. Bedford and J. Smillie [2, 3, 4], E. Bedford, M. Lyubich and J. Smillie [1], S. Friedland and J. Milnor in [10], J. Fornæss and N. Sibony [9], among others.

Complex Hénon maps are obtained as a finite composition of maps of the form $(y, p(y) - bx)$ where p is a polynomial of degree at least two and $b \in \mathbb{C}^*$. As in the case of rational maps in the one–dimensional context, complex Hénon maps have a well–defined Julia set J. This set is a compact invariant set that contains the support of the unique measure of maximal entropy. (See [2].) We will denote the support of the measure of maximal entropy by J^*.

A significant open question in the study of complex Hénon maps is whether $J = J^*$. Bedford and Smillie [2] show that if f is uniformly hyperbolic on J, then $J = J^*$. Further, Fornæss [8] shows that if f is uniformly hyperbolic on J^*, and is not volume preserving, then $J = J^*$. In the setting of complex Hénon maps, hyperbolicity is the natural generalization of the expansiveness on the Julia set for rational maps on $\overline{\mathbb{C}}$.

We recall that dominated splitting is a weaker version of hyperbolicity. In this setting, it is reasonable to look the property of dominated splitting, to verify the possible hyperbolicity. Recently in our work [21], was given several equivalences of uniform hyperbolicity, under the hypothesis of dominated splitting in the Julia set. Then a natural question appear:

Question: *When the set J/J^* has dominated splitting?.*

In the context of real and complex one–dimensional mapping, this phenomena is already known: for one–dimensional endomorphism, one obstruction to hyperbolicity is the presence of critical points (points with zero derivative) in the limit set. In fact, in the real context, Mañé showed that smooth and generic

(Kupka–Smale) one–dimensional endomorphisms without critical points are either hyperbolic or conjugate to an irrational rotation (see [16]). So we could say that, for generic smooth one–dimensional endomorphisms, any compact invariant set is hyperbolic if, and only if, it does not contain critical points. In the complex case (rational maps), it is well known that the Julia set J is hyperbolic if, and only if, J is disjoint of the post critical set. We recall that in dimension one, hyperbolicity and dominated splitting is the same notion.

Our main goal: is to introduce the dynamical obstruction to dominated splitting for two–dimensional biholomorphisms in a complex manifolds. This allow us to introduce one notion of **critical point** for complex Hénon maps, that capture many of the dynamical properties of their one–dimensional counterpart.

One notion of critical point on surfaces, was introduced by E. Pujals and F. Rodrigues Hertz in their work [18]. They works with systems that are dissipative in a compact invariant set without sink periodic point. The main result of [18] state that C^2–generically a systems has dominated splitting, if and only if, the set of critical point is empty. From Theorem B of Pujals–Sambarino in [19], the authors of [18] conclude that: *Generically, an invariant set is either an hyperbolic set or an normally hyperbolic closed curve which dynamics is conjugate to an irrational rotation if and only if the set of critical points is empty.* We remark that in [18], the authors performs the proof of their main result, using Theorem B on [19]. Later, S. Crovisier in [7], was give one prove of the main result on [18], independent of the Pujals–Sambarino's Theorem.

We will make a first presentation of our main result, in the context of complex Hénon maps. To introduce the notion of critical point we look for the projective action of the derivative of the map. More precisely, let f be a dissipative complex Hénon map, i.e. $|\det(df_x)| = b < 1$. Let F_x be the Möbius transformation induced by df_x. We denote the spherical norm of the derivative of F_x at the point $\xi \in \overline{\mathbb{C}}$ by $||F'_x(\xi)||$.

Let $b < \beta_+ \leq \beta_- < 1$ and $\beta = (\beta_-, \beta_+)$. We say that $x \in J$ is a β–**critical point** if there exists a direction ξ such that $||(F_x^{\pm n})'(\xi)|| \geq \beta_{\pm}^{\pm n}$, for each $n \geq 0$. We denote the set of all β–critical points by $\mathrm{Crit}(\beta)$. The preceding definition assert that a point is critical, if there exist a (projective) direction that is expanded (in norm) to the past by the action of F, and is not very contracted to the future. We recall that this definition generalize in the complex case, the notion of critical point in [18], that turn, and quoting the words of [18] authors, "...(a critical set)...goes back to the seminal studies done for Hénon attractor in [5]". Also recall that this definition is adapted to the dissipative context.

Our main result can be now stated as follows:

Main Theorem. *Let f be a dissipative complex Hénon map, with $|\det(df_x)| = b < 1$. Then J has dominated splitting if and only if for every $\beta = (\beta_-, \beta_+)$ where $b < \beta_+ \leq \beta_- < 1$, the set $\mathrm{Crit}(\beta)$ is an empty set.*

This Theorem is consequence of a more general version of this result, stated for complex linear cocycles, namely **Theorem A**. An hypothesis necessary both in the surfaces, and Theorem A version, is the absence of sinks. Since Julia set only contain periodic saddle points, this hypothesis not appear in the statement

of Main Theorem. For now, there are not a Pujals–Sambarino's Theorem in the two–dimensional complex case. The way to prove Theorem A is adapt in our context the main ideas of Crovisier on [7]. However, since the definition of critical point in [18], and our definition are distinct, the adaptation of this ideas, have several differences with the original version.

In this point, we explain several properties relatively to the critical set (See Subsection 6.2). Firstly, the critical set is a compact set. Introducing a partial order in the set of indexes β (we say that $\beta \geq \alpha$ if and only if $\beta_{\pm}^{\pm 1} \geq \alpha_{\pm}^{\pm 1}$), we have the monotony contention: if $\beta \geq \alpha$ then $\mathrm{Crit}(\beta) \subseteq \mathrm{Crit}(\alpha)$. Also, under change of (hermitian) metric and conjugation, critical point are preserved, maybe after of a finite but bounded iterates to the past or to the future. The critical set is far from dominated/hyperbolic sets and Pesin's Blocks. We recall that in the polynomial case, positive iterates of a critical point can still be critical, but not all element of the whole positive orbit is a critical point. This property also holds for our context: in the orbit of a critical point, there exists a *"distinguished"* critical point that is the last critical point (last in the point of view of the positive orbit). Critical point is not a regular point in the Osceledets sense. Finally, orbit of a tangencie between the stable and the unstable manifold of a periodic point, contain a critical point.

The paper is organized as follows:

In Section 2, we state result and give the tools related to complex linear cocycles (the action of df in the tangent of J) and projective cocycles (the action of F in the spherical bundle of J).

Section 3 is devoted to state the notion of dominated splitting for linear cocycles. We give several tools in terms of the projective cocycle, that are equivalent to the existence of dominated splitting.

In Section 4, we define formally the notion of critical point and state the general version of our Main Theorem (Theorem A). Also we state the notions necessaries to prove our Theorem A.

Section 5, is devoted to proving Theorem A.

Section 6, we study several properties of critical points. More over, in subsection 6.5, we conjecture (for complex Hénon maps) the existence of another "critical points" outside of J, in order to obtain a two dimensional counterpart for the classical one dimensional Theorem about rational maps: *If the post–critical is disjoint from the Julia set, them the Julia set is hyperbolic.*

2 Preliminaries

2.1 Bundles and Cocycles

Let X be a topological compact space. We let TX denote a locally trivial vector bundle of complex dimension 2 over X. We denote by T_z the fiber of TX at $z \in X$, and denote by $pr : TX \to X$ the natural projection. A **linear cocycle** $A : TX \to TX$ is a continuous isomorphism in the category of the vector bundles. More precisely, A is continuous and there exists $f : X \to X$ an homeomorphism such that $A_z = A|T_z : T_z \to T_{f(z)}$ is a complex isomorphisms. We say that the homeomorphism f is the **base** of the cocycle A.

Given a vector bundle PX we define its **projective bundle** as the set $\mathbb{P}(X) = \cup_{z \in X} \{z\} \times \mathbb{CP}^1(T_z)$. The projective bundle is a bundle which fiber is the Riemann sphere $\overline{\mathbb{C}}$. Denote the canonical projections by $p : TX^* \to \mathbb{P}(X)$, where $TX^* = TX \setminus \{\text{the zero section}\}$. In what follows, we denote by $\overline{\mathbb{C}}_z$ the fiber of $\mathbb{P}(X)$ at $z \in X$. Given A a linear cocycle with base id, p_A denotes the map from TX^* to $\mathbb{P}(X)$ satisfying $p_A|_{T_z^*} = p \circ A_z$.

Similarly as in the vector bundle case, we say that $M : \mathbb{P}(X) \to \mathbb{P}(X)$ is a **projective cocycle**, if it is a continuous isomorphisms in the category of bundles with projective fibre. Given a linear cocycle A we can associate to it a projective cocycle M in a natural fashion as $M_z([v]) = [A_z v]$, where $[\cdot]$ denote the equivalence class in the projective space.

Given a non–negative integer l we define the iterate of the cocycle A by the equation

$$A_z^l = A_{f^{l-1}(z)} \circ \cdots \circ A_{f(z)} \circ A_z \text{ and } A_z^{-l} = A_{f^{-l}(z)}^{-1} \circ \cdots \circ A_{f^{-2}(z)}^{-1} \circ A_{f^{-1}(z)}^{-1},$$

and define $A^0 = Id$ by convention. In the same way we define the iterates M_z^l and M_z^{-l} for the projective cocycle.

We have well–defined an hermitian (a spherical) metric in the linear (projective) bundle. Let $TX \odot TX$ be the subset of $TX \times TX$ consisting of pairs (u, v) such that u and v are in the same fiber. An **hermitian metric** on TX is a continuous function $(\cdot|\cdot) : TX \odot TX \to \mathbb{C}$ such that $(\cdot|\cdot)|T_z \times T_z = (\cdot|\cdot)_z$ is an hermitian product in T_z. Since X is compact, there exists an hermitian metric on TX (cf. [12]). In what follows, we denote $||v||_z = (v|v)_z$.

We also have the following statement.

Definition 1. *The **spherical metric** in the projective bundle $\mathbb{P}(X)$, is the metric induced by the hermitian metric in TX.*

For see the formal construction of the previous definition, we suggest to the reader consult the Appendix A.

2.2 Oseledets Theorem

We say that a point $z \in X$ is a **regular point of** A, if the fiber T_z admits a splitting $T_z = E_z \oplus F_z$ of one dimensional complex subspaces, and numbers

$\lambda^-(x) \leq \lambda^+(z)$ satisfying

$$\lim_{n \to \pm\infty} \frac{1}{n} \log(||A_z^n u||) = \pm\lambda^-(z) \quad \text{and} \quad \lim_{n \to \pm\infty} \frac{1}{n} \log(||A_z^n v||) = \pm\lambda^+(z),$$

where $u \in E_z \setminus \{0\}$ and $v \in F_z \setminus \{0\}$. Recall that a set $S \subset X$ has **total probability in** X, if for every f-invariant measure μ, we have $\mu(S^c) = 0$.

Theorem 2.1. (Oseledets) *The set of regular points of A has total probability. Moreover, $z \mapsto E_z$ and $z \mapsto F_z$ are measurable subbundles and the functions $z \mapsto \lambda^{\pm}(z)$ are measurable.*

For a proof of Oseledets's Theorem in the setting of cocycles, see [20].

We denote the set consisting of all regular points of a cocycle A by $\mathcal{R}(A)$. The Oseledets's Theorem asserts that given an f–invariant measure μ, the set of regular points in the support of μ has total measure. Indeed, we have that $\mu(\mathcal{R}(A) \cap \mathrm{supp}\,(\mu)) = 1$. We denote the set $\mathcal{R}(A) \cap \mathrm{supp}\,(\mu)$ by $\mathcal{R}(A, \mu)$.

In the original work of Pujals and Rodriguez Hertz (see [18]), an important hypothesis is the absence of saddle periodic points. In our setting, we replace this hypothesis for our next notion.

Definition 2. *We say that a measure f–invariant μ is **partially hyperbolic**, if for any $x \in \mathcal{R}(A, \mu)$ the inequality $\lambda^-(x) < 0 \leq \lambda^+(x)$ hold.*

*We also say that f **has no attractors** (in the broad sense) if all f–invariant measure is partially hyperbolic.*

2.3   The Multiplier

In the studies of rational maps in the Riemann sphere, an important tool to describe the dynamics near a periodic point, is the notion of multiplier. By Böcher's Theorem, the dynamic in a neighborhood of the periodic point is (via conjugation) given by the dynamics of the map $w \mapsto \lambda w$, where λ is called the multiplier of the point. In many cases, we are interested in the norm of the multiplier.

For a point $z \in \overline{\mathbb{C}}$ which is not periodic, it is possible to define a similar tool as the multiplier, using the spherical metric.

Definition 3. *Let $U \subset \overline{\mathbb{C}}$ be an open set and $R : U \to \overline{\mathbb{C}}$ be an holomorphic map. We define the **norm of the multiplier of R at the point** z, as the spherical norm of the derivative of R at the point z. That is,*

$$||R'(z)|| = \sup \left\{ \frac{||R'(z)\xi||_{R(z)}}{||\xi||_z} : \xi \in T_z\overline{\mathbb{C}} \right\}, \tag{1}$$

where $|| \cdot ||_z$ denote the spherical norm in $T_z\overline{\mathbb{C}}$.

Under the identification $T_z\overline{\mathbb{C}}$ with $\mathbb{C}$, an explicit expression for the spherical metric is

$$||\xi||_z = \frac{2|\xi|}{1+|z|^2}. \tag{2}$$

Thus, it is not difficult to see that

$$||R'(z)|| = |R'(z)| \cdot \frac{1+|z|^2}{1+|R(z)|^2}. \tag{3}$$

An important result is the following.

Proposition 2.1. *Let $z_0 \in \overline{\mathbb{C}}$ and R be a rational map. Also let f and h two isometries in the Riemann sphere such that $f(z) = h(R(z)) = 0$. Then*

$$||R'(z)|| = |(h \circ R \circ f^{-1})'(0)|.$$

Proof. Since f is an isometry in the Riemann sphere, for each $v \in T_w\overline{\mathbb{C}}$ we have that $||v||_w = ||f'(w)v||_{f(w)}$, and the same equality holds for h.

Thus we have that,

$$\frac{2}{1+|z|^2} = ||1||_z = ||f'(z)||_0 = 2|f'(z)|$$

and

$$\frac{2}{1+|R(z)|^2} = ||1||_{R(z)} = ||h'(R(z))||_0 = 2|h'(R(z))|,$$

therefore we conclude that

$$|(h \circ R \circ f^{-1})'(0)| = |(h'(R(z))| \cdot |R'(z)| \cdot |f'(z)|^{-1} = \frac{1}{1+|R(z)|^2} \cdot |R'(z)| \cdot (1+|z|^2).$$

$\square$

In the following lemma, we give an explicit formula to calculate the norm of the multiplier, for a Möbius transformation.

Lemma 2.1. *Let M be a Möbius transformations given by*

$$M(\xi) = \frac{a\xi + b}{c\xi + d},$$

then

$$||M'(z)|| = \frac{|\delta|}{||Av_z||^2} \tag{4}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

v_z is an unitary vector in $\mathbb{C}^2$ whit $[v_z] = z$, and $\delta = \det(A)$.

Proof. From equation (3), we have that

$$||M'(z)|| = |M'(z)| \cdot \frac{1 + |z|^2}{1 + |M(z)|^2} = \frac{|\delta|}{|cz + d|^2} \frac{1 + |z|^2}{1 + |M(z)|^2}.$$

If we take $v_z = (v_z^1, v_z^2) \in \mathbb{C}^2$ an unitary vector such that $z = v_z^1/v_z^2$, then

$$||M'(z)|| = |\delta| \cdot \frac{1 + |z|^2}{|az + b|^2 + |cz + d|^2}$$

$$= |\delta| \cdot \frac{1 + \left|\frac{v_z^1}{v_z^2}\right|^2}{\left|a\frac{v_z^1}{v_z^2} + b\right|^2 + \left|c\frac{v_z^1}{v_z^2} + d\right|^2}$$

$$= |\delta| \cdot \frac{|v_z^1|^2 + |v_z^2|^2}{|av_z^1 + bv_z^2|^2 + |cv_z^1 + dv_z^2|^2}$$

$$= \frac{|\delta|}{||Av_z||^2}.$$

$\square$

The equation (4), hallow define the norm of a multiplier for a projective cocycle M in terms of his linear cocycle related with them, and the hermitian metric considered in the fibre bundle.

Definition 4. *Let A be a linear cocycle on TX and M the natural projective cocycle related with them. Given $\xi \in \mathbb{P}(X)$ with $\xi \in \overline{\mathbb{C}}_z$ we define the* **norm of the multiplier of M at the point** $\xi \in \overline{\mathbb{C}}_z$ *by*

$$g(1, \xi) = g(1, \xi, M) = \frac{|\det(A_z)|}{||A_z v_\xi||_{f(z)}^2} \tag{5}$$

and similarly, the norm of the multiplier of M^n at the point ξ is defined by

$$g(n, \xi) = g(1, \xi, M^n) = \frac{|\det(A_z^n)|}{||A_z^n v_\xi||_{f^n(z)}^2} \tag{6}$$

where v_ξ is choose unitary and such that $[v_\xi] = \xi$.

Remark 1. *It follows from equation (6) that $g(n + m, \xi) = g(n, M^n \xi) \cdot g(m, \xi)$ for each $n, m \in \mathbb{Z}$. This prove is elementary and will be left to the reader.*

In the following subsection we justify the Definition 3, but is not essential for the rest of this work, and may be skipped.

2.4 Some Remarks

In the remainder of this section, we explain with detail the motivations for Definition 3. The experienced reader can skip directly to the next section.

First, we recall the formal definition of **multiplier** for a rational map at a fixed point.

Definition 5 (**Multiplier**). *Let R be a rational function on the Riemann sphere $\overline{\mathbb{C}}$, and let $z \in \overline{\mathbb{C}}$ be a fixed point of R.*

i) *If $z \in \mathbb{C}$ we define the multiplier of R in the point z by $R'(z)$, and is denoted by $\lambda(z, R)$.*

ii) *If $z = \infty$ we choose a Möbius map f such that $f(\infty) \in \mathbb{C}$, and it is defined $\lambda(\infty, R) = \lambda(f(\infty), f \circ R \circ f^{-1})$.*

Note that in the preceding definition the value of $\lambda(z, R)$ when $z \in \mathbb{C}$ remains *invariant under conjugation* by a Möbius transformations. It follows that $\lambda(\infty, R)$ is well–defined.

Remember that a Möbius transformation T is an isometry in the Riemann sphere whit the spherical metric, if and only if we can write

$$T(w) = T_{a,b}(w) = \frac{\overline{b}w + a}{-\overline{a}w + b},$$

with a and b complex number and $|a|^2 + |b|^2 = 1$. Note that if we write $z = a/b$ (and $z = \infty$ if $b = 0$) then $T(0) = z$. Since T is an isometry, we conclude that $T(\infty) = z^*$ where z^* is the antipodal point of z, that is $z^* = -1/\overline{z}$. Denote the set consisting of all isomorphisms in the Riemann sphere with the spherical metric by $\mathrm{Isom}(\overline{\mathbb{C}})$.

In this point, we want to extend the notion of multiplier for the case in which z is not a fixed point. The next proposition stay that this extension can not be done as expected.

Proposition 2.2. *Let $D \subset \overline{\mathbb{C}}$ be a topological disc and $R : D \to \overline{\mathbb{C}}$ be an holomorphic map. Let $z \in D \mapsto f_{i,z}, h_{i,z} \in \mathrm{Isom}(\overline{\mathbb{C}})$ with $i = 1, 2$, continuous maps such that $f_{i,z}(z) = h_{i,z}(R(z)) = 0$. If we define $F_{1,z}(w) = h_{1,z} \circ R \circ f_{1,z}^{-1}(w)$ and $F_{2,z}(w) = h_{2,z} \circ R \circ f_{2,z}^{-1}(w)$ in some neighborhood of zero, then there exists a unique continuous function $\xi : D \to \mathbb{S}^1$ such that*

$$F'_{1,z}(0) = \xi(z) F'_{2,z}(0). \tag{7}$$

Proof. If we write $f_{i,z} = T_{a_i(z), b_i(z)}$ and $h_{i,z} = T_{c_i(z), d_i(z)}$, then $f_{1,z} \circ f_{2,z}^{-1}(w) = \xi_1(z) \cdot w$ and $h_{2,z} \circ h_{1,z}^{-1}(w) = \xi_2(z) \cdot w$, where $\xi_1(z) = \zeta_1(z)/\overline{\zeta}_1(z)$, $\xi_2(z) = \zeta_2(z)/\overline{\zeta}_2(z)$, $\zeta_1(z) = a_1(z)\overline{a}_2(z) + \overline{b}_1(z)b_2(z)$, and $\zeta_2(z) = c_2(z)\overline{c}_1(z) + \overline{d}_2(z)d_1(z)$. Taking $\xi(z) = (\xi_1(z))^{-1} \cdot \xi_2(z)$ we obtain the Proposition. $\square$

Note that the number $\widetilde{\lambda}(z, R) := F'_{1,z}(0)$ is the multiplier in the fixed point case. Nevertheless, the preceding Proposition establishes that $\widetilde{\lambda}(z, R)$ depend of the isometries considered in the "conjugation", but $|\widetilde{\lambda}(z, R)|$ is independent. So, is reasonable to call this number as "the norm of the multiplier". Since Proposition 2.1 establishes that that $|\widetilde{\lambda}(z, R)|$ is equal to the spherical norm of $R'(z)$, we decided to accept this terminology in the Definition 3.

3 Dominated Splitting and Hyperbolic Projective Cocycles

The main goal of this section is to characterize the notion of dominated splitting for a linear cocycle, in terms of his action in the projective bundle. We introduce the notion of hyperbolic projective cocycle, that roughly speaking, are those cocycles that present the same dynamics as a hyperbolic Möbius transformation. In Theorem 3.1, we prove that a linear cocycle has dominated splitting if and only if his projective cocycle is hyperbolic. Moreover, in the same Theorem we estate that the continuity of the section it is not necessary to obtain domination.

We recall the notion of dominated splitting for linear cocycles.

Definition 6. *We say that a cocycle $A : TX \to TX$ has **dominated splitting** if there exist an A–invariant splitting $TX = E \oplus F$ where E and F are one-dimensional complex planes, and $l \geq 1$ such that*

$$||A^l_z|_{E_z}|| \cdot ||A^{-l}_{f^l(z)}|_{F_{f^l(z)}}|| < \frac{1}{2}.$$

Recall that if A has dominated splitting, then the A–invariant splitting $TX = E \oplus F$ is continuous. The following is a classical result that establishes equivalences properties with dominated splitting.

Proposition 3.1. *Let A be a linear cocycle on a vector bundle TX. Then the following statements are equivalent:*

1. *The cocycle $A : TX \to TX$ has dominated splitting.*

2. *There exist an A–invariant splitting $TX = E \oplus F$ where E and F are one–dimensional complex planes, a constant $0 < \lambda < 1$, and a $C > 0$ such that*
$$||A^n_z|_{E_z}|| \cdot ||A^{-n}_z|_{F_{f^n(z)}}|| \leq C\lambda^n$$
for every $z \in X$ and all $n > 0$.

3. *There exist a splitting $TX = E \oplus F$ where E and F are one–dimensional complex planes, a constant $l \geq 0$, and cone fields $K(\alpha, E)$ and $K(\beta, F)$, where*
$$K(\alpha, E_z) = \big\{ u + v \in E_z \oplus F_z \ : \ ||u|| \leq \alpha ||v|| \big\}$$

and

$$K(\beta, F_z) = \big\{ u + v \in E_z \oplus F_z \ : \ ||v|| \le \beta ||u|| \big\},$$

A^l–*invariant, that is*

$$A^{-l}_{f^l(z)}(K(\alpha, E_{f^l(z)})) \subset K(\alpha, E_z)^\circ, \ \ A^l_z(K(\beta, F_z)) \subset K(\beta, F_{f^l(z)})^\circ$$

and

$$||A^l_z|_{K(\alpha, E_z)}|| \cdot ||A^{-l}_{f^l(z)}|_{K(\beta, F_{f^l(z)})}|| < \frac{1}{2}{}^1$$

where $K^\circ = \mathrm{int}(K) \cup \big\{0\big\}$.

Remark 2.   *1. It follows from the preceding definition that a linear cocycle A has dominated splitting if and only if A^n has dominated splitting, for all $n \ge 2$.*

2. Note that item 3 in the preceding establish that it is possible to get the invariant splitting given in item 1 and item 2, by the expressions

$$\widetilde{E}_z = \bigcap_{n \ge 0} A^{-n}_{f^n(z)}(K(\alpha, E_{f^n(z)})) \ \ and \ \ \widetilde{F}_z = \bigcap_{n \ge 0} A^n_{f^{-n}(z)}(K(\beta, F_{f^{-n}(z)})). \quad (8)$$

The main idea, is note that the invariants splitting determines (in each fibre) two specials points in the sphere, and the cones fields represent disks in the riemann sphere that are asymptotically contracted/expanded by the projective cocycle. We explain this remark in the subsequent paragraphs.

Definition 7.   *1. We say that a section $\sigma \in \Gamma(X, \mathbb{P}(X))$ is M–**invariant** if $M(\sigma(z)) = \sigma(f(z))$.*

*2. We say that a section σ is **contractive** if it is M–invariant and there exist constant $C > 0$ and a $0 < \lambda < 1$, such that $g(n, \sigma(z)) \le C\lambda^n$ for every $z \in X$ and all $n \ge 1$. Similarly, we say that a section is **expansive** if this is contractive for M^{-1}.*

*3. We say that a cocycle M is **hyperbolic** if there exist two disjoint sections τ and σ in $\Gamma(X, \mathbb{P}(X))$ (i.e., $\tau(z) \ne \sigma(z)$ for every $z \in X$) such that τ is expansive and σ is contractive.*

So we can state the following:

Theorem 3.1. *Let A be a linear cocycle on TX and M the projective cocycle on $\mathbb{P}(X)$ related with them. Then the following properties are equivalents:*

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[1]This condition appear for real cocycle of each dimension. In our case, complex two-dimensional dominated splitting is not necessary. To see this fact, see Proposition 3.3, and proof of Theorem 3.1, Claim 1, in the Subsection 3.5.

1. *The cocycle A has dominated splitting*

2. *The cocycle M is hyperbolic.*

3. *There exists σ a contractive section for M (equivalently there exists τ an expansive section).*

4. *There exist $C > 0$ and $\lambda > 1$ such that for every $z \in X$ there exists one direction $\tau_z \in \overline{\mathbb{C}}_z$ such that $g(n, \tau_z) \geq C\lambda^n$ for every $n > 0$ (equivalently $\sigma_z \in \overline{\mathbb{C}}_z$ such that $g(-n, \sigma_z) \geq C\lambda^n$ for every $n > 0$).*

Theorem 3.1 will be proved in Subsection 3.5. In the following subsections we explain a series of results necessaries for his proof.

3.1 Conjugation of Cocycles and Global Trivialization Bundle

A well known fact about holomorphics maps, is that topological (metrical) contraction of small disc around some point implies that the norm of its derivative is smaller than one and therefore also its multiplier is smaller than one. Since that projective cocycle is holomorphic in each fiber, to determinate if the norm of the multiplier is less to one in some point, it is suffices to determinate if this contract disc around this point. For that, it is natural to look for more simples cocycles which are conjugated to the initial one, and check if the new cocycle shrinks discs. The formal notion of conjugation is the following definition.

Definition 8. *Let $M, N : \mathbb{P}(X) \to \mathbb{P}(X)$ be two cocycles with $M = (f, M_*)$ and $N = (g, N_*)$. We say that M and N are* **conjugated** *if there exists a cocycle $H = (h, H_*) : \mathbb{P}(X) \to \mathbb{P}(X)$ where $h : X \to X$ is an homeomorphism such that $H \circ M = N \circ H$.*

The preceding definition state that we have simultaneously the conjugations $hf(z) = gh(z)$ and $H_{f(z)} M_z(\xi) = N_{h(z)} H_z(\xi)$.

In what follows, we will only work with projective cocycles with an invariant (global) section. Whit this hypothesis, the following result establish that the bundle is trivial.

Proposition 3.2. *If $\mathbb{P}(X)$ has a global section, then $\mathbb{P}(X)$ is isometrically equivalent to the trivial bundle $X \times \overline{\mathbb{C}}$.*

Proof. Let $\sigma \in \Gamma(X, \mathbb{P}(X))$ be a global section and E a splitting in TX associated with this direction. Let us take σ^* the global section associated with the direction $E^\perp$, then $\sigma^*(z)$ is the antipodal point of $\sigma(z)$ in the sphere $\overline{\mathbb{C}}_z$.

Claim: *For every $z \in X$ there exists a biholomorphism $H_z : \overline{\mathbb{C}}_z \to \overline{\mathbb{C}}$ such that is an isometry, $H_z(\sigma(z)) = 0$ and $H_z(\sigma^*(z)) = \infty$.*

Proof of Claim. First, let $\big\{ (U_i, \varphi_i) : i = 1, \ldots, n \big\}$ be a family of local charts of bundle such that $X = \cup_i U_i$. Take $v_i \in \Gamma(U_i, TX)$ a local sections where

$||v_i|| = 1$ and $v_i(z) \in E$. Let $L_z : T_z \to \mathbb{C}^2$ be the unique linear isometry such that $T_z(v_i(z)) = (1,0)$ and $\det(L_z) = 1$. The map L_z is unique because the only element of the group $SU(2, \mathbb{C})$ that fix the vector $(1,0)$ is the identity map. Since each v_i is unique and varies continuously, we conclude that $z \mapsto L_z$ is continuous.

Now, take $v_i^*(z) = L_z^{-1}((0,1)) \in \overline{\mathbb{C}}_z$. We conclude that $v_i^* \in \Gamma(U_i, TX)$, $||v_i^*|| = 1$, and $v_i^*(z) \in E^\perp$. Define the splitting

$$F_z = \big\{ v \in T_z \ : \ (v|v_i - v_i^*)_z = 0 \big\}.$$

It is easy to see that F_z is independent of the choice of v_i's and F is a continuous splitting. We conclude that F define a global section $\tau \in \Gamma(X, \mathbb{P}(X))$, this is, for any $u \in F_z$ we have $\tau(z) = [u]$.

Finally, we define H_z as the unique Möbius transformation such that $H_z(\sigma(z)) = 0$, $H_z(\sigma^*(z)) = \infty$, and $H_z(\tau(z)) = 1$; more precisely we define $H_z([v]) = [L_z(v)]$. This finishes the proof of the claim.

Continuing with the proof of the Proposition, if (U, ϕ) is a local of the bundle $\mathbb{P}(X)$, by continuity of the sections, the local expression in U of H_z is a continuous function. More precisely, there exist a continuous family $\widetilde{H} : U \times \overline{\mathbb{C}} \to U \times \overline{\mathbb{C}}$ with

$$\widetilde{H}_z(w) = \frac{a_z w + b_z}{c_z w + d_z}$$

where the maps $z \mapsto a_z, \ldots, z \mapsto d_z$ are continuous, $H_z = \widetilde{H}_z \circ \phi$, $\widetilde{H}_z \circ \widetilde{\sigma}(z) = 0$, and $\widetilde{H}_z \circ \widetilde{\sigma^*}(z) = \infty$, where $\widetilde{\sigma} = \sigma \circ \phi$ and $\widetilde{\sigma^*} = \sigma^* \circ \phi$. It follows that the function $H = (id, H_*)$ is an homeomorphism and an isometry in each fiber. $\square$

Remark 3. *After previous proposition we can assume that the bundle $\mathbb{P}(X)$ is in fact the trivial bundle $X \times \overline{\mathbb{C}}$. Moreover, given a section $\sigma \in \Gamma(X, \mathbb{P}(X))$ we can lift this section to the trivial bundle $X \times \mathbb{C}^2$ as a global section $v \in \Gamma(X, X \times \mathbb{C}^2)$ such that $||v|| = 1$ and if we write $v = (v_1, v_2)$ then $\sigma(z) = (z, [v_1(z) : v_2(z)])$; this helps us to find global expressions of the section in the projective bundle.*

With this remark we can define.

Definition 9. *A section v as in Remark 3, is called a **unitary lift** of the section σ.*

3.2 Equivalence of Contractive sections

We denote the unit disc in $\mathbb{C}$ by $\mathbb{D}$. Given $\xi \in \overline{\mathbb{C}}$ an $r > 0$, we denote the *ρ–ball with center at ξ and radius r* in the spherical metric by $B(\xi, r)$. For any isometry in the Riemann sphere L with $L(0) = \xi \in \overline{\mathbb{C}}$, the set $L(r\overline{\mathbb{D}})$ does not depend on L. We will denote this set by $D_r(\xi)$ and is called *disc of radius r*

centered at ξ. Moreover, we have that for any r, the disc $D_r(\xi)$ is equal to $B(\xi, \varepsilon)$ where ε satisfies the equation

$$\frac{r}{\sqrt{1+r^2}} = \sin\left(\frac{\varepsilon}{2}\right).$$

This last equation goes from the relation between the chordal and spherical metric (see for example, [6]).

Proposition 3.3. *Let* $\sigma \in \Gamma(X, \mathbb{P}(X))$ *be a M–invariant section. Then the following statement are equivalents:*

    i. The section σ is contractive.

    ii. There exist $0 < \eta < 1$ and $k > 0$ such that $g(k, \sigma(z)) < \eta$ for all z in X.

    iii. There exist $k > 0$ and $r > 0$ such that $M_z^k(\overline{D}_r(\sigma(z))) \subset D_r(\sigma(f^k(z)))$.

    iv. There exist $k > 0$ and $R > 0$ such that for all $0 < r \leq R$, $M_z^k(\overline{D}_r(\sigma(z))) \subset D_r(\sigma(f^k(z)))$.

Proof. Clearly (i) implies (ii). To see that (ii) implies (i), define

$$C_j = \sup\left\{g(j, \sigma(z)) \ : \ z \in X\right\}$$

for $j = 0, \ldots, k-1$. We conclude that for every $s \geq 0$

$$g(sk+j, \sigma(z)) = g(sk, \sigma(f^j(z)))g(j, \sigma(z)) \leq C_j\eta^s = C_j\eta^{-j/k}[\eta^{1/k}]^{sk+j} \leq C\lambda^{sk+j}$$

where $C = \sup\left\{C_j\eta^{-j/k} \ : \ j = 0, \ldots, k-1\right\}$ and $\lambda = \eta^{1/k} < 1$.

    Also it is clear that (ii) is equivalent with (iii) and that (iv) implies (ii) and (iii).

    To prove that (ii) implies (iv), we consider $v = (v_1, v_2)$ the unitary lift of σ. Consequently, if σ^* is the antipodal point of σ then $v^* = (\overline{v}_2, -\overline{v}_1)$ is an unitary lift of σ^*. We consider

$$B_z = \left(\begin{array}{cc} \overline{v}_2(z) & v_1(z) \\ -\overline{v}_1(z) & v_2(z) \end{array}\right) \tag{9}$$

and denote the Möbius transformation related with them by H_z. It follows that H_z is an isometry of the Riemann sphere.

    Now we take $N_z = H_{f(z)}^{-1} \circ M_z \circ H_z$. If we define the cocycles N and H by $N = (f, N_*)$ and $H = (id, H_*)$, then $H \circ N = M \circ H$ and that the null section $\xi_0 \equiv 0$ is N–invariant. We conclude that N_z has the form

$$N_z(\xi) = \frac{\xi}{\beta_z\xi + \alpha_z}.$$

Notice that

$$g(1, \xi_0) = g(1, \xi_0, N) = \left|\frac{1}{\alpha_z}\right|$$

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so from hypothesis there exist constants $0 < \eta < 1$ and $k \geq 0$ such that $g(k, \xi_0(z)) = |\alpha_z^k|^{-1} < \eta$ for every z in X. Consequently there exists a $R > 0$ such that for every $0 < r \leq R$ and $z \in X$, we have $g(k, \xi) \leq \eta$ for every $\xi \in D_r(\xi_0(z))$. It follows that there exist a constant $0 < \lambda < 1$ and a $C > 0$ such that $g(n, \xi) \leq C\lambda^n$ for every ξ in $D_r(\xi_0(z))$ and $n \in \mathbb{N}$. The previous observation implies that $|N^n(\xi)| < C\lambda^n r$ for every $\xi \in D_r(\xi_0)$ and $n \geq 1$, as required. $\qquad\square$

Corollary 3.1. *Let σ and $R > 0$ satisfying the item (iv) of the Proposition 3.3. Then for every $\xi \in D_r(\sigma(z))$ and $0 < r \leq R$, we have*

$$\lim_{n \to +\infty} \rho(M_z^n(\xi), M_z^n(\sigma(z))) = 0$$

where ρ is the spherical metric.

3.3   Expansive/Contractive Direction

In this subsection we explain some properties of the function g. The main goal is establish the uniqueness of direction asymptotically expansive. This fact will be used recurrently in this work.

Lemma 3.1. *If ξ_i with $i = 1, 2$ are two different directions in $\overline{\mathbb{C}}_z$ and u_i is an unitary vector that generate the direction ξ_i for $i = 1, 2$, then*

$$g(n, \xi_1)g(n, \xi_2) = \left(\frac{\sin(\angle(A_z^n u_1, A_z^n u_2))}{\sin(\angle(u_1, u_2))} \right)^2,$$

for any $n \in \mathbb{Z}$.

Proof. Let $x, y \in \mathbb{C}^2$ and denote the area of the polygon formed by the vertices $0, x, x + y$ and y by $\phi(x, y)$. Then we have the equality

$$\phi(x, y) = |x| \cdot |y| \cdot \sin(\angle(x, y)) = \sqrt{\det([x\ y]^* \cdot [x\ y])} = |\det([x\ y])|, \qquad (10)$$

where $[x\ y]$ is a column matrix and $[x\ y]^*$ denote its transposed conjugate. Then, it is easy to see that $\phi(Ax, Ay) = |\det(A)|\phi(x, y)$ for any linear map A in $\mathbb{C}^2$.

Then from equation (10) we have that

$$\left(\frac{\sin(\angle(A_z^n u_1, A_z^n u_2))}{\sin(\angle(u_1, u_2))} \right)^2 = \frac{\phi(A_z^n u_1, A_z^n u_2)^2 / |A_z^n u_1|^2 \cdot |A_z^n u_2|^2}{\phi(u_1, u_2)^2 / |u_1|^2 \cdot |u_2|^2} = \frac{|\det(A_z^n)|^2}{|A_z^n u_1|^2 \cdot |A_z^n u_2|^2}.$$

According to the equation (6) and the previous equality, it follows that

$$g(n, \xi_1)g(n, \xi_2) = \frac{|\det(A_z^n)|^2}{|A_z^n u_1|^2 \cdot |A_z^n u_2|^2} = \left(\frac{\sin(\angle(A_z^n u_1, A_z^n u_2))}{\sin(\angle(u_1, u_2))} \right)^2.$$

$\qquad\square$

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Lemma 3.2. *An expansive (contractive) direction is unique.*

Proof. Take $z \in X$ and two different directions that are expanded for the future $\xi_1, \xi_2 \in \overline{\mathbb{C}}_z$, that is, there exist $C > 0$ and $\lambda > 1$ such that $g(n, \xi_i) \geq C\lambda^n$ for each $n \geq 0$, and $i = 1, 2$. If u_i is an unitary vector that generate the direction ξ_i for $i = 1, 2$, we conclude that $\angle(u_1, u_2) > 0$. From the preceding Lemma, we have that

$$C^2 \lambda^{2n} \leq g(n, \xi_1) g(n, \xi_2) = \left(\frac{\sin(\angle(A_z^n u_1, A_z^n u_2))}{\sin(\angle(u_1, u_2))} \right)^2 < \frac{1}{\sin(\angle(u_1, u_2))}$$

which is a contradiction. For the case that we have expansion for the past, the same proof holds. $\square$

3.4 Module

A **double connected domain** in $\overline{\mathbb{C}}$ is a open connected set U such that its complement has two connected component. The definition of the module of a double connected domain is based in the following mapping theorem: *Every double connected domain U is biholomorphic to a ring domain of the form*

$$A(r_1, r_2) = \left\{ z \in \mathbb{C} : 0 \leq r_1 < |z| < r_2 \leq \infty \right\}$$

and is called a canonical image of U.

If $r_1 > 0$ and $r_2 < \infty$ for one canonical image of U, then the ratio of the radii r_2/r_1 is the same for all canonical images of U. Then the number

$$\mathrm{mod}(U) = \log\left(\frac{r_2}{r_1} \right)$$

determines the conformal equivalence class of U and is called the module of U. Otherwise, we define $\mathrm{mod}(U) = \infty$ and this happens if and only if at least one boundary component of U consists of a single point.

The following proposition will be crucial in the proof of Theorem 3.1.

Proposition 3.4. *Let $D_1, D_2, D_3, \ldots$ be conformal discs in $\overline{\mathbb{C}}$ such that for every $i \geq 1$ we have $\overline{D}_i \subset D_{i+1}$. If there exist a constant $\kappa > 0$ such that $\mathrm{mod}(\mathrm{int}(D_{i+1} \setminus \overline{D}_i)) \geq \kappa$, then the set $D = \cup_n D_n$ is biholomorphic to $\mathbb{C}$.*

See [17, 15] for details.

3.5 Proof of Theorem 3.1

We define the **stable set** at the point $\xi \in \overline{\mathbb{C}}_z$ of a cocycle M as the set

$$W^s(\xi) = \left\{ w \in \overline{\mathbb{C}}_z : \lim_{n \to \infty} \rho(M_z^n(w), M_z^n(\xi)) = 0 \right\}$$

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and the **local stable set of size** $\varepsilon > 0$ by

$$W_\varepsilon^s(\xi) = \{w \in W^s(\xi) \,:\, \rho(M_z^n(w), M_z^n(\xi)) < \varepsilon, \text{ for all } n \in \mathbb{N}\}.$$

The unstable set is defined in the same way, for the inverse cocycle M^{-1}.

We can write the stable (resp. unstable) set in terms of backward (resp. forward) iterates of the local stable (resp. local unstable) sets. In fact, given $\varepsilon > 0$ we have

$$W^s(\xi) = \bigcup_{n=0}^{\infty} M_z^{-n}\left(W_\varepsilon^s(M^n(\xi))\right) \text{ and } W^u(\xi) = \bigcup_{n=0}^{\infty} M_z^n\left(W_\varepsilon^u(M^{-n}(\xi))\right).$$

Also we have the following statement.

Lemma 3.3. *Let σ be a contractive section for M. Then there exist constants $k \geq 0$ and $r > 0$, such that*

$$W^s(\sigma(z)) = \bigcup_{t \geq 0} M_z^{-tk}\left(D_r(\sigma(f^{tk}(z)))\right).$$

Proof. From Corollary 3.1 it follows that $D_r(\sigma(f^{tk}(z)))$ is the local stable set. Since this is uniformly contractive by the cocycle, we have our Lemma. $\square$

Proof of Theorem 3.1. Our proof goes through a series of claims.

Claim 1: *A linear cocycle A has dominated splitting if and only if the cocycle $M([v]) = [Av]$ is hyperbolic.*

Proof of Claim 1. Suppose that A has dominated splitting.

First, let $\{(U_i, \varphi_i) : i = 1, \ldots, l\}$ be a family of local chart of the bundle TX such that $X = \cup_i U_i$. Let $u_i \in \Gamma(U_i, E)$, and let $v_i \in \Gamma(U_i, F)$ such that $\|u_i(z)\|_z = \|v_i(z)\|_z = 1$ for every $z \in U_i$ and $i = 1, \ldots, l$. For each $z \in U_i$ define

$$\tau(z) = (z, [u_i(z)]) \text{ and } \sigma(z) = (z, [v_i(z)]).$$

It follows that both σ and τ are well–defined continuous global section in TX, which are M–invariant.

Now, let $K(\beta, F)$ be an A^l–invariant cone field and denote the set $p_I(K(\beta, F_z))$ by D_z. Without loss of generality, we can assume that $l = 1$ (See Remark 2.) We recall that D_z is a closed conformal disc, that is, a biholomorphism image of the closed unitary disc. Note that:

1. $\sigma(z) \in D_z$,

2. $M_z(D_z) \subset \text{int}(D_{f(z)})$,

3. $\sigma(z) = \bigcap_{n \geq 0} M_{f^{-n}(z)}^n(D_{f^{-n}(z)})$.

Item 1 and item 2 it follows directly from definition. Item 3 it follows from equation (8). Note also that item 2, implies that $\cap_{n=0}^{m} M_{f^{-n}(z)}^{n}(D_{f^{-n}(z)}) = M_{f^{-m}(z)}^{m}(D_{f^{-m}(z)})$. From compactness of X there exists $r > 0$ such that $D(\sigma(z), r) \subset D_z$. We conclude that there exists $k > l$ such that

$$M_{f^{-k}(z)}^{k}(D(\sigma(f^{-k}(z)), r)) \subset \mathrm{int}(D(\sigma(z), r)).$$

Finally, since the splitting varies continuously (and consequently the cone fields) and by compactness of X, we conclude that the constant k is independent of the choice of the point $z \in X$. By Proposition 3.3, we conclude that σ is contractive. An argument similar applied to τ and M^{-1}, implies the hyperbolicity of M.

Conversely we suppose that M is hyperbolic.

First, we denote by E_z and F_z the sets $p_I^{-1}(\tau(z)) \cup \{0\}$ and $p_I^{-1}(\sigma(z)) \cup \{0\}$ respectively. Clearly this define a A-invariant splitting. Take u and v unitary lifts of τ and σ respectively (See Definition 9.) Remember that the unitary lift are element of the section of the trivial vector bundle $X \times \mathbb{C}^2$.

Now, we construct a hyperbolic cocycle N conjugated with M. Write $u = (u_1, u_2)$ and $v = (v_1, v_2)$. Define H_z as the Möbius transformation related with the matrix

$$B_z = \begin{pmatrix} u_1(z) & v_1(z) \\ u_2(z) & v_2(z) \end{pmatrix}, \qquad (11)$$

define $N_z = H_{f(z)}^{-1} \circ M_z \circ H_z$, and also define the cocycles N and H by $N = (f, N_*)$ and $H = (id, H_*)$. Clearly $H \circ N = M \circ H$. We assert that the cocycle N is hyperbolic. In fact, note that by construction, the sections ξ_∞ (resp. ξ_0) that associates at each point the point at infinity (resp. the zero point), are N–invariant. Consequently, we have that $N_z(\xi) = \lambda_z \xi$. Since M is hyperbolic, then M^k shrinks small closed disc around σ and expands small disc around τ for some $k \geq 0$ (see Proposition 3.3). Thus a similar phenomena holds for N^k. We conclude that $|\lambda_z^k|$ is less than one for every $z \in X$. Moreover, the compactness of X allows to take this constant uniformly in z, that is, there exist $0 < \eta < 1$ such that $|\lambda_z^k| < \eta$. Since $g(k, \xi_0(z)) = |\lambda_z^k| < \eta$ and $g(k, \xi_\infty(z)) = |\lambda_z^{-k}| > \eta^{-1} > 1$ it follows the hyperbolicity of N.

Finally, let $\{(U_i, \varphi_i)\}$ be a finite family of local chart of the bundle TX such that $X = \cup_i U_i$. Let $\widetilde{u}_i, \widetilde{v}_i \in \Gamma(U_i, TX^*)$ such that $\widetilde{u}_i \in E_z$, $\widetilde{v}_i \in F_z$ and both are unitary. Then we have

$$p_I(\widetilde{u}_i) = \tau = (id, [u]) \quad \text{and} \quad p_I(\widetilde{v}_i) = \sigma = (id, [v]).$$

Take $z \in U_i$ with $f^n(z) \in U_j$ and write $A_z^n \widetilde{u}_i(z) = k_z^n \widetilde{u}_j(f^n(z))$, $A_z^n \widetilde{v}_i(z) = l_z^n \widetilde{v}_j(f^n(z))$ and $\widetilde{A}_z^n = (\varphi_j)_z^{-1} \circ A_z^n \circ (\varphi_i)_z$ then

$$B_{f^n(z)}^{-1} \widetilde{A}_z^n B_z = \begin{pmatrix} k_z^n & 0 \\ 0 & l_z^n \end{pmatrix}$$

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is the matrix related with the Möbius transformation N_z^n. We conclude that $\lambda_z^n = k_z^n/l_z^n$. Now taking $u' = su_i(z)$ in E_z and $v' = tv_j(f^n)$ in $F_{f^n(z)}$ with $|s| = |t| = 1$, then

$$||A_z^n u'|| \cdot ||A_{f^n(z)}^{-n} v'|| = ||A_z^n u_i(z)|| \cdot ||A_{f^n(z)}^{-n} v_j(f^n(z))|| = |k_z^n| \cdot |l_z^n|^{-1} = |\lambda_z^n| \leq C\lambda^n$$

for $n \geq 1$, so A has dominated splitting. $\qquad\square$

Claim 2: *Let σ be a contractive section for M (resp. τ be a expansive section). Then for every $z \in X$, $W^s(\sigma(z))$ (resp. $W^u(\tau(z))$) is biholomorphic to $\mathbb{C}$.*

Proof of Claim 2. Take k and r as in Lemma 3.3 and define

$$D_t = M_z^{-tk} \left(D_r(\sigma(f^{tk}(z))) \right).$$

It is clear that $D_{t-1} \subsetneq D_t$ and the function M_z^{tk} maps biholomorphically $D_t \setminus D_{t-1}$ on

$$A_t = D_r(\sigma(f^{tk}(z))) \setminus M_{f^{(t-1)k}(z)}^k(D_r(\sigma(f^{(t-1)k}(z)))),$$

so the module $\mathrm{mod}(D_t \setminus D_{t-1})$ and $\mathrm{mod}(A_t)$ are equal. We take $0 < \eta < 1$ uniformly in X such that $g(k,\sigma) \leq \eta$. It follows that $\mathrm{mod}(A_t) \geq \log(1/\eta)$. By Corollary 3.4, the claim is proves. $\qquad\square$

Claim 3: *A projective cocycle M is hyperbolic if and only if at least one of the following equivalent conditions hold:*

    a. There exist a section that is a contraction.

    b. There exist a section that is an expansion.

Proof of Claim 3. We need only show that (b) imply (a), because the reciprocal direction follows using the inverse cocycle.

From the previous Claim, we have that $\overline{\mathbb{C}}_z \setminus W^s(\sigma(z)) = \{\tau(z)\}$. Since W^s varies continuously and is M–invariant, it follows that τ is a M–invariant section. By the definition of τ, it follows that small disc around of τ are contracted uniformly by M^{-1}, and it follows that τ is an expansion. $\qquad\square$

Claim 4: *If a projective cocycle M satisfy item 4 of this Theorem, then there exist a section that is a expansion.*

Proof of Claim 4. First, we denote by u_z some unitary vector that define the direction τ_z. Since we have uniqueness of an expansive direction (Lemma 3.2), we conclude that $M(\tau_{f^{-1}(z)}) = \tau_z$.

Now, let $z_n \to z$ in X, then $\tau_{z_n} \to \tau_z$. In fact, by compactness there exists some adherence point for the sequence $(\tau_{z_n})_n$, named $\tau' \in \overline{\mathbb{C}}_z$ that is expansive for the future. From Lemma 3.2, it follows that τ' is equal to τ_z, and hence we have that the function $z \mapsto \tau_z$ is continuous.

Finally, from the hypothesis we have that τ is an expansion. Then, from Claim 3 and Claim 1 we have that X has dominated splitting. $\qquad\square$

This conclude the proof of Theorem 3.1. $\qquad\square$

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4 Critical Points and Main Theorem

To give a precise definition of critical point and state out Theorem A, firstly we introduce some technical notations. Let Δ be the set defined by

$$\Delta = \{\beta = (\beta_-, \beta_+) : 0 < \beta_+ \leq \beta_- < 1\}.$$

In Δ we have a partial order. In fact, let $\alpha, \beta \in \Delta$, we say that $\beta \geq \alpha$ if and only if $\beta_\pm^{\pm 1} \geq \alpha_\pm^{\pm 1}$.

Definition 10.   *1. Let $\beta \in \Delta$ and $n_- \leq 0 \leq n_+$ integers. We say that $x \in X$ is a β–**critical point at the times** (n_-, n_+) (for the linear cocycle A), if there exist a direction $\xi_x \in \overline{\mathbb{C}}_x$ such that for every $n \geq 0$ we have that $g(\pm n, M^{n\pm}\xi_x) \geq \beta_\pm^{\pm n}$. The direction ξ_x will be called **critical direction**.*

*2. We say that x is a β–**critical point** if this is a β–critical point at the times $(0,0)$. We denote the set of all β–critical point by $\mathrm{Crit}(\beta)$.*

*3. We say that y is a β–**critical value** if y is a β–critical point for the linear cocycle A^{-1}. We denote the set of all β–critical value by $\mathrm{CVal}(\beta)$.*

Remark 4. *It follows easily from the previous definition that if $\alpha, \beta \in \Delta$ such that $\beta \geq \alpha$ then $\mathrm{Crit}(\beta) \subset \mathrm{Crit}(\alpha)$.*

Note that from Lemma 3.2, the critical direction is unique. In Section 6 we explain in details a series of properties of critical points. For the moment, we have the following property.

Remark 5. *The previous definition say that*

$$\mathrm{Crit}(\beta) = \{x \in X \,:\, \exists \xi_x \;\; such \; that \;\; g(\pm n, \xi_x) \geq \beta_\pm^{\pm n}, \;\; n \geq 0\}$$

and

$$\mathrm{CVal}(\beta) = \{x \in X \,:\, \exists \xi_x \;\; such \; that \;\; g(\mp n, \xi_x) \geq \beta_\pm^{\mp n}, \;\; n \geq 0\}$$

In order to state the Main Theorem, we need the following.

Definition 11. *Given $0 < b < 1$, we say that X is b–**asymptotically dissipative** (for a cocycle A) if there exists a positive constant $C > 0$ such that for every $z \in X$, $|\det(A_z^n)| \leq Cb^n$ for every $n \geq 0$.*

We recall that: given a cocycle $A = (f, A_*)$, we say that f has no attractors, if all measure f–invariant is partially hyperbolic (See Definition 2). Our Main Theorem is the following.

Theorem A. *Let $A = (f, A_*)$ be a linear cocycle over X such that f has no attractors. Assume that X is b–asymptotically dissipative for the cocycle A. Then A has dominated splitting if and only if for each $\beta \in \Delta$ which $\beta_+ > b$, the set $\mathrm{Crit}(\beta)$ is an empty set.*

In the remainder of this section, we state several tools necessaries to prove the Main Theorem.

4.1 Blocks of Domination

Definition 12. *Given $\alpha \in (0,1)$ and $\gamma > 0$, we define the the blocs of domination as the sets*

$$\gamma H^{\pm}(\alpha) = \left\{ z \in X : \exists\, \xi \in \overline{\mathbb{C}}_z \;\; such\; that\;\; g(\pm n, \xi) \geq \gamma \alpha^{-n},\, \forall\, n \geq 0 \right\},$$

and the sets

$$\gamma \mathring{H}^{\pm}(\alpha) = \left\{ z \in X : \exists\, \xi \in \overline{\mathbb{C}}_z \;\; such\; that\;\; g(\pm n, \xi) > \gamma \alpha^{-n},\, \forall\, n \geq 0 \right\}.$$

Remark 6.   *1. When $\gamma = 1$, we denote the set $\gamma H^{\pm}(\alpha)$ (resp. $\gamma \mathring{H}^{\pm}(\alpha)$) by $H^{\pm}(\alpha)$ (resp. $\mathring{H}^{\pm}(\alpha)$).*

   2. It is easy to see that if $0 < \alpha \leq \alpha'$ then $\gamma H^{\pm}(\alpha) \subseteq \gamma H^{\pm}(\alpha')$ and $\gamma \mathring{H}^{\pm}(\alpha) \subset \gamma \mathring{H}^{\pm}(\alpha')$.

The next Theorem establishes condition for the existence of blocks of domination. We prove them in subsection 6.3.

Theorem 4.1. *Let $A = (f, A_*)$ be a linear cocycle over X such that f has no attractors. Then for any $1 > \beta > b$ and $1 \leq \gamma < b^{-1}\beta$ the blocks of domination $\gamma H^{+}(\beta)$ and $\gamma^{-1} H^{-}(\beta)$ are not empty compact sets. Moreover, the sets*

$$X_0^{+} = \cup_{n \in \mathbb{Z}} f^n(\gamma H^{+}(\beta)) \;\; and \;\; X_0^{-} = \cup_{n \in \mathbb{Z}} f^n(\gamma^{-1} H^{-}(\beta))$$

have total measure for any invariant measure ν with support in X.

In [18], the authors introduce the notion of critical point in terms of the block of domination. We decided introduce this notion independently of them. In subsection 6.1, we explain in details this relations.

4.2 Pliss's Lemma

The following Lemma is a remarkable result known as Pliss lemma, which is frequently used in this paper.

Lemma 4.1. (Pliss's Lemma) *Given $0 < \gamma_1 < \gamma_0$ and $a > 0$, there exist $N_0 = N_0(\gamma_0, \gamma_1, a)$ and $\delta_0 = \delta_0(\gamma_0, \gamma_1, a) > 0$ such that for any sequences of numbers $(a_l)_{l=0}^{n-1}$ with $n > N_0$, $a^{-1} < a_l < a$ and $\prod_{l=0}^{n-1} a_l \geq \gamma_0^n$ we have that: if*

$$Ht = \left\{ 0 \leq k < n \,:\, \forall\, k < s < n,\; we\; have\; that\; \prod_{l=k+1}^{s} a_l \geq \gamma_1^{s-k} \right\}, \qquad (12)$$

then $\#Ht \geq n \cdot \delta_0$.

Remark 7. *When $k \in Ht$, k is called a* **hyperbolic time**.

As a corollary, we have the following result.

Corollary 4.1. *Given $0 < \gamma_1 < \gamma_0$ there exist N_0 and δ_0 positive constants such that: If for $z \in X$ there exists $\xi \in \overline{\mathbb{C}}_z$ satisfying $g(n, \xi) \geq \gamma_0^n$ (resp. $g(-n, \xi) \geq \gamma_0^n$) for some $n \geq N_0$, then there exists $0 \leq j < n$ such that $n - j > n\delta_0 - 1$ and*

$$g(i, M^j(\xi)) \geq \gamma_1^i \quad \text{for every} \ \ 0 < i \leq n - j,$$

(resp. $g(-i, M^{-j}(\xi)) \geq \gamma_1^i \quad$ for every $\quad 0 < i \leq n - j$).

Proof. From Pliss's Lemma, taking $a_l = g(1, M^l(\xi))$ for $l = 0, \ldots, n - 1$ then $g(n, \xi) = \prod_{l=0}^{n-1} g(1, M^l(\xi)) \geq \gamma_0^n$. Let k_0 be the lowest hyperbolic time. We have that $n - k_0 \geq n\delta_0$, and for every $k_0 < s < n$

$$\gamma_1^{s-k_0} \leq \prod_{l=k_0+1}^{s} g(1, M^l(\xi)) = g(s - k_0, M^{k_0+1}(\xi)).$$

Hence it is enough to take $j = k_0 + 1$, and we have the corollary. $\qquad\square$

4.3 Criteria for Domination

Now we present a criteria for the existence of dominated splitting that is essential in the proof of our Main Theorem.

Theorem 4.2. (Criteria for Domination) *Let X be a b–asymptotically dissipative for a cocycle $A = (f, A_*)$ such that f has no attractor, and let $1 > \beta > b$.*

If there exist $k_0, m_0 > 0$ such that for all $z \in X$, there exists one direction $\xi_z \in \overline{\mathbb{C}}_z$ satisfying that

$$g(k_0, M^m(\xi_z)) \leq \beta^{-k_0}, \quad \text{for every} \ \ m > m_0; \tag{13}$$

then X has dominated splitting.

To prove the Criteria for Domination we need of the following.

Proposition 4.1. *Suppose that there exist $k_1, m_1 > 0$ and $\gamma < 1$ such that for any $z \in X$, there exists one direction $\xi_z \in \overline{\mathbb{C}}_z$ such that*

$$g(k_1, M^m(\xi_z)) < \gamma, \quad \text{for every} \ \ m > m_1; \tag{14}$$

then X has dominated splitting.

Proof. Fix $z_0 \in X$ and denote by $\xi_0 = M^{m_1}(\xi_{z_0})$ and $\xi_t = M^t(\xi_0)$, then we have that $g(k_1, \xi_t) < \gamma$ for every $t \geq 0$. Let us take, for $j = 0, \ldots, k_1 - 1$

$$C_j = \sup\{g(j, w) \ : \ w \in \overline{\mathbb{C}}_z \, , \ z \in X\},$$

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it follows that

$$g(nk_1 + j, \xi_0) = g(j, \xi_0)g(nk_1, \xi_j) \leq C_j \gamma^n \leq C\lambda_0^{nk_1+j},$$

where $\lambda_0 = \gamma^{1/k_1} < 1$ and $C_0 = \sup\{C_j\gamma^{-j/k_1} : j = 1, \ldots, k_1 - 1\}$.

To end, for every $z \in X$ let us take $z_0 = f^{-m_1}(z)$ and $\sigma_z = M^{m_1}(\xi_{z_0})$, it follows that $g(-n, \sigma_z) \geq C\lambda^n$, where $C = C_0^{-1}$ and $\lambda = \lambda_0^{-1}$. From Theorem 3.1 item 4, we conclude the domination. $\square$

A fundamental tool to prove the Criteria for Domination, is the following lemma. This establish that if there exists one direction that is neither contracted nor expanded for the future, then the largest Lyapunov exponent in the omega limit of this point is negative. Our version is a stronger version of Main Lemma in [18], but enough to conclude what we want.

Lemma 4.2. (Criteria of Negative Exponent) *Let $1 > \beta_2, \beta_1 > b$, let $x \in X$ and let ξ_x be a direction in $\overline{\mathbb{C}}_x$. Suppose that there exist constants $n_0, m_0 \in \mathbb{N}$ such that*

i) $\omega(x)$ is b–asymptotically dissipative,

ii) $\beta_2^n \leq g(n, \xi_x)$ for every $n \geq n_0$.

iii) $g(n, M^m\xi_x) \leq \beta_1^{-n}$ for every $m > m_0$ and $n \geq n_0$.

Then $\omega(x)$ supports a measure μ which large exponent is negative, and so, ν is not a partially hyperbolic measure.

Proof. We may assume that $\omega(x)$ only support partially hyperbolic measures, and then the large exponent is not negative. We take $n_k \to \infty$ such that the limit

$$\mu = \lim_{k \to \infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \delta_{M^i(\xi_x)}$$

there exists. We denote the support of the measure μ by $\widehat{K}$. Then $\widehat{K}$ is a compact set of TX and his projection $K = pr(\widehat{K}) \subset \omega(x)$ is the support (in X) of the measure

$$\mu' = \lim_{k \to \infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \delta_{f^i(x)}$$

that is the projection of μ in the first coordinate.

Since μ' is a f–invariant measure, we have that for any $z_0 \in \mathcal{R}(A, \mu')$ and $w \in \overline{\mathbb{C}}_{z_0}$, the limit

$$\lim_{k \to \infty} \frac{1}{n_k} \log\left(g(n_k, w)\right) = \lim_{k \to \infty} \frac{1}{n_k} \left(\log(|\det A_{z_0}^{n_k}|) - 2\log(|A_{z_0}^{n_k}w|)\right) \tag{15}$$
$$= \lambda^+(z_0) + \lambda^-(z_0) - 2\lambda(w)$$

there exists, where $\lambda(w)$ is the Lyapunov exponent associated with the direction w. We denote the limit given in the equation (15) by $I(z_0, w)$. Since $\lambda(w) = \lambda^\pm(z_0)$, we conclude that $I(z_0, w)$ take only the values $\lambda^+(z_0) - \lambda^-(z_0)$ or $\lambda^-(z_0) - \lambda^+(z_0)$.

Since that $z_0 \in \omega(x)$, there exists a sequences $(m_k)_k$ such that $f^{m_k}(x) \to z_0$. By passing to subsequence if necessary, there exists $w_0 \in \overline{\mathbb{C}}_z$ such that $M^{m_k}(\xi_x) \to w_0$. We conclude that (z_0, w_0) is a point of $\widehat{K}$.

By item (iii) of this Lemma, we have that

$$I(z_0, w_0) \leq -\log(\beta_1). \tag{16}$$

Moreover, this inequality is true for every $(z, w) \in \widehat{K}$, with $z \in \mathcal{R}(A, \mu')$.

On the other hand, we remark that $\lambda^-(z_0) \leq \log b < 0 \leq \lambda^+(z_0)$ and $\lambda^-(z_0) - \lambda^+(z_0) \leq \lambda^-(z_0) + \lambda^+(z_0) \leq \log b$. Hence either $I(z_0, w_0) \leq \log b$ or $I(z_0, w_0) \geq -\log b$. If we suppose that the second inequality holds, then $I(z_0, w_0) \geq -\log b > -\log \beta_1$ that contradict the equation (16). We conclude that for every $(z, w) \in \widehat{K}$ with z a regular point in the Oseledets sense, the limit $I(z, w)$ is equal to

$$\lim_{k \to \infty} \frac{1}{n_k} \log(g(n_k, w)) = \lambda^-(z) - \lambda^+(z) \leq \log(b).$$

Claim. $\mu(\widehat{K} \cap pr^{-1}(\mathcal{R}(A, \mu'))) = 1$.
Proof of the Claim. The Ergodic Decomposition Theorem assert that: There exists a set Σ of full probability in $\mathbb{P}(X)$ such that for all $(z, w) \in \Sigma$ the limit

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{M^j(z,w)} = \mu_{(z,w)}$$

is an ergodic measure, and that for all $h \in \mathcal{L}^1(\mathbb{P}(X), \mu)$ we have

$$\int \left(\int h d\mu_{(z,w)} \right) d\mu = \int h d\mu.$$

In particular, the projection in the first coordinate

$$\mu'_z = pr \circ \mu_{(z,w)} = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(z)}$$

is ergodic and

$$\int \left(\int \widetilde{h} d\mu'_z \right) d\mu = \int \widetilde{h} d\mu'$$

where $\widetilde{h} \in \mathcal{L}^1(X, \mu')$.

On the other hand, note that our claim is true for ergodic measures. Recall also that, since the measures μ'_z is ergodic, the Lyapunov exponents are invariant

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functions which are constant in the support of the measures μ'_z. Now, the set $R(z) = pr^{-1}(\mathcal{R}(A, \mu'_z))$ is invariant by the projective cocycle, so has $\mu_{(z,w)}-$measure 0 or 1, but

$$\mu_{(z,w)}(\log(g)) = \mu_{(y,t)}(\log(g)) = \lambda^-(y) - \lambda^+(y)$$

for $(y,t) \in R(z)$ $\mu_{(z,w)}-$a.e.. The preceding implies that $\mu_{(z,w)}(R(z)) \neq 0$, and so is equal to 1, which completes the proof of the claim.

Continuing with the proof of the Lemma, applying Birkhoff Ergodic Theorem to the function $\phi = \log(g)$, μ–a.e. $(z, w) \in \widehat{K}$, there exist the limit

$$\widetilde{\phi}(z, w) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi \circ M^j(w) = \lambda^-(z) - \lambda^+(z),$$

hence it follows that

$$\log(b) \geq \int \widetilde{\phi} d\mu(z, w) = \int \phi d\mu(z, w) = \lim_{k \to \infty} \frac{1}{n_k} \log(g(n_k, \xi_x)) \geq \log \beta_2,$$

which is a contradiction, because $\beta_2 > b$. $\qquad\square$

Proof of Criteria for Domination. Let β_0 be a constant such that $1 > \beta_0 > \beta > b$. From Proposition 4.1, to obtain domination we need only to prove that there exist positive integers k_1 and m_1, such that for every z, there exists one direction ξ_z satisfying $g(k_1, M^m \xi_z) < \beta_0^{k_1}$ for all $m > m_1$, as in equation (14).

If not, we have that every pair $k_1, m_1 \in \mathbb{N}$, there exists $z \in X$ such that for every $\xi \in \overline{\mathbb{C}}_z$ we have that $g(k_1, M^m \xi) \geq \beta_0^k$ for some $m > m_0$. We conclude, in particular, that for every k there exist $z_k \in X$ and $m_k > k$ such that $g(k, M^{m_k} \xi_k) \geq \beta_0^k$, where ξ_k satisfies the equation (13) in the statement of this Proposition.

We take $1 > \beta_0 > \beta_2 > \beta$. Applying Corollary 4.1 to the constants β_0 and β_2, we conclude that there exists a sequence $(r_k)_{k \geq 1}$ with $k - r_k \to \infty$ such that

$$g(s, M^{r_k} \xi_k) \geq \beta_2^s, \text{ for every } 0 < s \leq k - r_k.$$

Taking z and ϖ as an accumulations point of $(f^{r_k}(z_k))_k$ and $(M^{r_k}(\xi_k))_k$, respectively, it follows that

$$\beta_2^n \leq g(n, \varpi), \text{ for every } n > 0.$$

On the other hand, since for every k and $m \geq m_0$ we have that $g(k_0, M^m(\xi_k)) \leq \beta^{-k_0}$, so we conclude that there exists a constant $C > 0$ such that for every $k > 0$, $n \geq 0$ and $m \geq m_0$ we have $g(n, M^m(\xi_k)) \leq C\beta^{-n}$. Taking $\beta > \beta_1 > b$ and n_0 large enough we obtain that for $n \geq n_0$, we have that $g(n, M^m(\xi_k)) \leq \beta_1^{-n}$. Passing to the limit we conclude that for all $m \geq m_0$ and $n \geq n_0$

$$g(n, M^m(\varpi)) \leq \beta_1^{-n}.$$

In this point, we are in the hypothesis of the Criteria of Negative Exponent, hence there exists an invariant measure that is not partially hyperbolic supported in X, which is a contradiction. $\qquad\square$

5   Proof of Main Theorem

This section is based in the ideas of Sylvain Crovisier (see [7]), for the proof of the same result in the context of C^2 generic dimorphism in compact manifolds (vide, [18]). Our exhibition presents significant changes compared with that of Silvan, among others, we have a different definition of critical point than [18]. Now we present a notion that allows to prove the Main Theorem.

Definition 13. *Given* $1 > \beta_0 > 0$, *we say that a projective cocycle* M *satisfies the property* $P(\beta_0)$ *if there exist* $k_0 > 0$, *such that for every* $k > k_0$ *there exit* $x_k \in X$, $\xi_k \in \overline{\mathbb{C}}_{x_k}$ *and* $m_k \geq 0$ *such that:*

1. $g(-n, \xi_k) \geq \beta_0^{-n}$, *for every* $1 \leq n \leq k$,

2. $g(k, M^{m_k}(\xi_k)) \geq 1$.

The next Lemma, will be prove in the next Section.

Lemma 5.1. *Let* $(a_n)_{n \in \mathbb{Z}} \subset \mathbb{R}$ *be a sequence. Assume that there exist* $n_0 \leq 0 \leq n_1$ *and* $-\infty < \delta_+ \leq \delta_- < 0$ *such that:*

a) $a_n - a_{n_0} \geq (n - n_0)\delta_-$, *for all* $n \leq n_0$,

b) $a_n - a_{n_1} \geq (n - n_1)\delta_+$, *for all* $n \geq n_1$.

Then there exists $N \in [n_0, n_1]$ *such that* $a_{\pm n + N} - a_N \geq \pm n\delta_\pm$ *for all* $n \geq 0$.

Proposition 5.1. *Let* $1 > \beta_0 > 0$. *If the projective cocycle* M *satisfies the property* $P(\beta_0)$, *then for every* $\beta \in \Delta$ *which* $(\beta_0, \beta_0) \geq \beta$ *the set* $\mathrm{Crit}(\beta)$ *is not empty.*

Proof. First, let $(\beta_0, \beta_0) \geq \beta = (\beta_-, \beta_+)$. We recall that $\beta_+ \leq \beta_0 \leq \beta_-$, and that for each $0 \leq n \leq k$ we have that $g(-n, \xi_k) \geq \beta_0^{-n} \geq \beta_-^{-n}$.

Now, we will apply the Corollary 4.1. Let $k > 0$, $\gamma_0 = 1$, $\gamma_1 = \beta_+$ and let n_0 and $\delta_0 > 0$ be the numbers given by this Corollary. If we choose $s > n_0$ such that $s\delta_0 - 1 > k$, since that $g(s, M^{m_s}\xi_s) \geq 1$, then there exist $0 \leq j < s$ such that $s - j > s\delta_0 - 1 > k$ and

$$g(i, M^{m_s + j}\xi_s) \geq \beta_+^i, \quad \text{for every} \quad 0 < i \leq s - j.$$

Therefore taking $y_k = x_s$, $v_k = \xi_s$ and $n_k = m_s + j$, we obtain that for every $k > 0$, there exist $y_k \in X$, $v_k \in \overline{\mathbb{C}}_{y_k}$ and $n_k \geq 0$ such that, for all $0 < n \leq k$,

$$g(-n, v_k) \geq \beta_-^{-n} \text{ and } g(n, M^{n_k}v_k) \geq \beta_+^n.$$

Define $n_0 = 0$, $n_1 = n_k$, $\delta_\pm = \log(\beta_\pm)$ and

$$a_n = \begin{cases} n\delta_- & , n < -k \\ \log(g(n, v_k)) & , -k \leq n \leq n_k + k \\ (n - n_k)\delta_+ + a_{n_k} & , n > n_k \end{cases}.$$

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It is not difficult to see that we are in the hypothesis of Lemma 5.1, and hence there exists $-l_k \in [0, n_k]$ such that $a_{\pm n - l_k} - a_{-l_k} \geq \pm n \delta_{\pm}$ for all $n \geq 0$. From the construction of sequence $(a_n)_n$ we can conclude that $g(\pm n, M^{l_k} \xi_x) \geq \beta_{\pm}^{\pm n}$ for all $0 \leq n \leq k$.

Finally, if we take $z_k = f^{l_k}(y_k)$ and $\omega_k = M^{l_k}(v_k)$, we have that for each $0 < n \leq k$

$$g(-n, \omega_k) \geq \beta_-^{-n} \quad \text{and} \quad g(n, \omega_k) \geq \beta_+^n.$$

To end, take (z, ω) an adherence point of (z_k, ω_k), and we have that for $n \geq 0$

$$g(-n, \omega) \geq \beta_-^{-n} \quad \text{and} \quad g(n, \omega) \geq \beta_-^n,$$

then $\mathrm{Crit}(\beta)$ is nonempty as asserted. $\qquad\square$

We denote by $\mathrm{supp}\,(X)$ the closed subset of X that support all measure f–invariant, i.e.,

$$\mathrm{supp}\,(X) = \overline{\cup \{ \mathrm{supp}\,(\nu) : \nu \text{ is } f\text{-invariant } \}}.$$

Lemma 5.2. *If $1 > \beta_0 > b$, then $\mathrm{supp}\,(X) \subset \omega(H^-(\beta_0))$.*

Proof. Any point in the support of an invariant measure ν is approximated by regular points. By the proof of Theorem 4.1[2], any $x \in \mathcal{R}(A, \nu)$ has infinitely many iterates in $H^-(\beta_0)$. The previous remark and the Poincaré recurrence theorem implies that

$$\mathrm{supp}\,(\nu) \subset \omega(H^-(\beta_0)) = \overline{\bigcup_{z \in H^-(\beta_0)} \omega(z)},$$

and this implies that $\mathrm{supp}\,(X) \subset \omega(H^-(\beta_0))$. $\qquad\square$

Lemma 5.3. *If there exist $1 > \beta_0 > b$ such that the property $P(\beta_0)$ is not satisfied, then the set $\mathrm{supp}\,(X)$ has dominated splitting.*

Proof. As the property $P(\beta_0)$ is not satisfied, then there exists $k > 0$ such that for every $x \in X$ and $v \in \overline{\mathbb{C}}_x$ both $g(-n, v) < \beta_0^{-n}$ for $1 \leq n \leq k$, or $g(k, M^m v) < 1$ for every $m \geq 1$. In particular, for points $x \in H^-(\beta_0)$ with critical direction ξ, the first inequality can not holds, then

$$g(k, M^m \xi) < 1. \tag{17}$$

We claim that $\omega(H^-(\beta_0))$ has dominated splitting. From equation (17), we have that for every $x \in H^-(\beta_0)$ (and denoting his critical direction by ξ), $g(k, M^m \xi) < \beta_0^{-k}$. Let $z \in \omega(x)$ and $(m_l)_l$ be a sequence of positives integers goes to infinity such that $f^{m_l}(x) \to z$. By passing to a subsequence if necessary,

[2]See subsection 6.3

there exists a direction $\xi_z \in \overline{\mathbb{C}}_z$ such that $M^{m_l}(\xi) \to \xi_z$. From continuity of g we conclude that: for every $x \in H^-(\beta_0)$, and $z \in \omega(x)$ there exist $\xi_z \in \overline{\mathbb{C}}_z$ satisfying

$$\beta_0^{-k} > g(k, M^{m+m_l}\xi) = g(k, M^m(M^{m_l}\xi)) \to g^k(M^m(\xi_z)).$$

Hence this implies that $\omega(H^-(\beta_0))$ satisfies the hypothesis of the Criteria for Domination. In particular $\operatorname{supp}(X)$ has dominated splitting. $\square$

We can rewrite the item 4 of Theorem 3.1 in terms of blocs of dominations. Moreover, we can state.

Lemma 5.4. *Let $\Lambda \subseteq X$ be a compact f–invariant. Then the linear cocycle A has dominated splitting in Λ if and only if there exist $\gamma, \beta^* > 0$ such that $\Lambda \subset \gamma \mathring{H}^-(\beta)$, and if and only if $\Lambda \subset \gamma \mathring{H}^+(\beta)$, where $\beta \in (\beta^*, 1)$ is arbitrarily.*

Proof. Let $T\Lambda = E \oplus F$ be a dominated splitting. By Proposition 3.1, we have that there exist constants $C > 0$ and $0 < \lambda < 1$ such that $g^{-n}(F_z) \geq C^{-1}\lambda^{-n}$ and $g^n(E_z) \geq C^{-1}\lambda^{-n}$. We may take λ minimal with this property. Then taking $C^{-1} = \gamma$ and $\beta^* = \lambda$ we obtain the necessary.

The suffices direction is immediate from item 4 of Theorem 3.1. $\square$

Proposition 5.2. *If $\operatorname{supp}(X)$ has dominated splitting but X does not have dominated splitting, then there exists β^* (minimal) such that the property $P(\beta_0)$ is satisfied on X, for every $\beta_0 \in (\beta^*, 1)$.*

Proof. Let β^* as in Lemma 5.4. Since that X does not have a dominated splitting and denying the Criteria for Domination, follows that for every positive integer k, there exists a point $x_k \in X$, and an integer $m > 0$ such that for every direction $\omega \in \overline{\mathbb{C}}_{x_k}$ we have that

$$g(k, M^m\omega) \geq 1. \tag{18}$$

On the other hand, the set α-limit of x_k, that we denote by $\alpha(x_k)$, supports an f–invariant measure, hence there exists $z_0 \in \alpha(x_k) \cap \operatorname{supp}(X)$. From the preceding Corollary, there exists one direction $\xi_0 \in \overline{\mathbb{C}}_{z_0}$ such that

$$g(-n, \xi_0) > \gamma(\beta')^{-n}, \quad \text{for every} \quad n \geq 1,$$

where $\beta' \in (\beta^*, 1)$.

Take $\beta^* < \beta' < \beta_0 < 1$ and k fixed.

Let $(n_t)_t \nearrow \infty$ such that $f^{-n_t}(x_k) \to z_0$. For every positive integer s, we can find some neighborhood $U_s \subset \mathbb{P}(X)$ of ξ_0 such that for every $\xi \in U_s$, holds that $g(-n, \xi) \geq \gamma(\beta')^{-n}$ for all $1 \leq n \leq s$. If we take t great enough, $f^{-n_t}(x_k)$ is inside of the projection in X of neighborhood U_s. Hence, there exists $\xi_s \in \overline{\mathbb{C}}_{f^{-n_t}(x_k)}$ such that $g(-n, \xi_s) \geq \gamma(\beta')^{-n}$ for all $1 \leq n \leq s$. Note that

for s great enough we have that $g(-s, \xi_s) \geq \beta_0^{-s}$, hence we are in the hypothesis of Corollary 4.1.

We conclude that, we can find s and l_s such that $s - l_s > k$ and

$$g(-n, M^{-l_s}\xi_s) \geq \beta_0^{-n}, \quad \text{for every} \quad 0 < n \leq s - l_s,$$

in particular, writing $v_k = M^{-l_s}(\xi_s)$, we conclude that

$$g(-n, v_k) \geq \beta_0^{-n}, \quad \text{for every} \quad 0 < n \leq k.$$

From equation (18), we have that there exist m_k such that $g(k, M^{m_k}v_k) \geq 1$, so the property $P(\beta_0)$ is satisfied. $\qquad \square$

With this in mind, we can prove one direction of Main Theorem. In the next subsection, we present their proof in the other direction.

Proof of Main Theorem: *If X does not have dominated splitting, then for each $\beta \in \Delta^+$ with $\beta_+ > b$ we have $\text{Crit}(\beta) \neq \emptyset$.*

We claim that there exist $1 > \beta_0 > b$ such that the property $P(\beta_0)$ holds.

In fact, if we assume that for each $1 > \beta_1 > b$ not holds, then from Lemma 5.3 we conclude that $\text{supp}(X)$ has dominated splitting. Since X does not have dominated splitting, from Proposition 5.2 there exists β^* such that for all $1 > \beta_1 > \max(b, \beta^*)$ the property $P(\beta_1)$ holds, that is a contradiction, hence as was claimed, the property $P(\beta_0)$ holds.

Proposition 5.1 imply that $\text{Crit}(\beta) \neq \emptyset$ for each $(\beta_0, \beta_0) \geq \beta$.

$\qquad \square$

5.1 Critical Pair

Now we work to proof the opposite direction of the Main Theorem. For this, we use the fact that for every critical point, there exists a critical value intrinsically linked with him. This is the notion of critical pair that we introduce in the following paragraph.

Definition 14. *Let $\beta \in \Delta$. We say that a pair $(x, y) \in X \times X$ is a β–critical pair if:*

1. *$x \in \text{Crit}(\beta)$, with critical direction ξ,*

2. *$y \in \text{CVal}(\beta)$, with critical direction ϖ,*

3. *there exist a sequence of positive integer l_k such that*

$$f^{l_k}(x) \to y \quad and \quad M^{l_k}\xi \to \varpi.$$

It follows directly of the previous definition the following Proposition.

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Proposition 5.3. *If X has dominated splitting, then X does not have a β–critical pair.*

Proof. If A have dominated splitting $TX = E \oplus F$, then the angle of the invariant splitting is great of some $\alpha > 0$. If (x, y) is critical pair, the direction F_x is defined by ξ, and E_y is defined by ϖ, but by the third condition on the previous definition we have that $M^{l_k}(F_x) \to E_y$, and this say that $F_y = E_y$; a contradiction. $\qquad\square$

The following Proposition, related each β–critical point with a β–critical value.

Proposition 5.4. *Let $\beta \in \Delta$. Then for every β–critical point x, there exists a β–critical value y, such that the pair (x, y) is a β–critical pair.*

Remark 8. *Given a critical point x, the critical value y is not, a priori, uniquely defined, can be occurs that for different critical values y and y', makes (x, y) and (x, y') critical pairs.*

Only remains to proof the Proposition 5.4. For this, we need of the next lemma.

Lemma 5.5. *Let X be a b–asymptotically dissipative such that every f–invariant measure is partially hyperbolic. Let $1 > \beta_1, \beta_2 > b$, let $x \in X$, and let $\xi_x \in \overline{\mathbb{C}}_x$. If $g(n, \xi_x) \geq \beta_2^n$ for each $n \geq 0$, then for every $k > 0$ there exists m_k such that*

$$g(n, M^{m_k}\xi_x) \geq \beta_1^{-n}$$

for all $0 \leq n \leq k$.

Proof. Let $\beta_1 > \beta_0 > b$. The proof goes through the following claim.

Claim. *For every n' and m' positive, there exist $l \geq n'$ and $m \geq m'$ such that*

$$g(l, M^m \xi_x) \geq \beta_0^{-l}.$$

Proof of Claim. If the previous assertion not holds, we have that there exist n', m' such that for all $l \geq n'$ and $m \geq m'$, $g(l, M^m \xi_x) < \beta_0^{-l}$. Since $g(n, \xi_x) \geq \beta_2^n$, for each $n \geq n'$ we are in the hypothesis of the Criteria of Negative Exponents, that is a contradiction. $\qquad\square$

From the preceding, we conclude that for each $n' \geq 1$, there exist $l \geq n'$ and $m \geq 1$ such that $g(l, M^m \xi_x) \geq \beta_0^{-l}$.

Now, we will apply the Corollary 4.1. Let $k > 0$ fixed. Let $\gamma_0 = \beta_0^{-1}$ and $\gamma_1 = \beta_1^{-1}$, and let n_0 and δ_0 be the numbers given by this Corollary. We take $n' > n_0$ such that $n'\delta_0 - 1 > k$. Since $g(l, M^m \xi_x) \geq \beta_0^{-l}$, there exists $0 \leq j < l$ such that $l - j \geq l\delta_0 - 1 \geq n'\delta_0 - 1 > k$ and that

$$g(i, M^{m+j}\xi_x) \geq \beta_1^{-i}$$

for each $0 < i \le l - j$. Therefore taking $m_k = m + j$, we obtain that

$$g(n, M^{m_k}\xi_x) \ge \beta_1^{-n}$$

for each $0 \le n \le k$, as asserted. $\qquad\square$

Proof of Proposition 5.4: Let $x \in \mathrm{Crit}(\beta)$ with critical direction ξ_x. Since $\beta_+ > b$ and $g(n, \xi_x) \ge \beta_+^n$, from Lemma 5.5 it follows that for each positive k, there exists m_k such that

$$g(n, M^{m_k}\xi_x) \ge \beta_-^{-n}$$

for all $0 \le n \le k$.

On the other hand, we have that for each $n \ge 0$

$$g(-n, \xi_x) \ge \beta_-^{-n} > 1 > \beta_+^n.$$

We define $n_0 = -m_k$, $n_1 = 0$, $\delta_\pm = -\log(\beta_\pm)$ and

$$a_n = \begin{cases} \log(g(-n, \xi_x)) & n_0 - k \le n < +\infty \\ (n - n_0)\delta_- + a_{n_0} & n < n_0 - k \end{cases}.$$

It is not difficult to see that we are in the hypothesis of Lemma 5.1, and hence there exists $-l_k \in [-m_k, 0]$ such that $a_{\pm n - l_k} - a_{-l_k} \ge \pm n \delta_\pm$ for all $n \ge 0$. From the construction of sequence $(a_n)_n$ we can conclude that $g(\mp n, M^{l_k}\xi_x) \ge \beta_\pm^{\pm n}$ for all $0 \le n \le k$.

From compactness, and taking a subsequence if necessary, there exist $y \in X$ and $\varpi \in \overline{\mathbb{C}}_y$ such that

$$f^{l_k}(x) \to y \quad \text{and} \quad M^{l_k}\xi \to \varpi$$

and therefore $y \in \mathrm{CVal}(\beta)$. $\qquad\square$

Now we will conclude the proof of Main Theorem.

Proof of Main Theorem: *If* $\mathrm{Crit}(\beta) \ne \emptyset$ *with* $\beta_+ > b$, *then* X *does not have Dominated Splitting:*
If there exist critical point, then there exist a critical pair, so by Proposition 5.3, X does not have dominated splitting. $\qquad\square$

6   Properties of the Critical Point

In this section we explain in details a series of properties referring to critical points. Initially, we state notions, that represent equivalences with the notion of critical points.

Definition 15. *We say that $x \in X$ is a β–**post–critical point of order** $N \in \mathbb{Z}^+$, if there exists $n \in \mathbb{Z}$ with $|n| \leq N$ such that $f^n(x) \in \mathrm{Crit}(\beta)$.*

Note that, a post–critical point of order 0, is a critical point.

In the definition above, when n is negative, it is more natural to replace the word "post–critical" by "pre–critical". To avoid overloading the language, we choose the terminology post–critical given the sense that this point is an iterate (positive or negative) of a critical point.

Our following result, explain that really we have only post–critical points.

Theorem 6.1. *If $x \in X$ be a β–critical point at the times (n_-, n_+), then x is a β–post–critical point of order $N = \max\{n_+, -n_-\}$.*

To prove this Theorem, we need of the Lemma 5.1.

Proof of Lemma 5.1. The proof goes through the following claim.

Claim: *Let $(a_n)_{n\in\mathbb{Z}} \subset \mathbb{R}$ be a sequence and there exist $n_0 \leq 0 \leq n_1$ and $-\infty < \delta_+ \leq \delta_- < 0$ such that:*

   i) $a_n \geq (n - n_0)\delta_-$, *for all* $n \leq n_0$,

   ii) $a_n \geq (n - n_1)\delta_+$, *for all* $n \geq n_1$.

Then there exists $N \in [n_0, n_1]$ such that $a_{\pm n + N} - a_N \geq \pm n \delta_\pm$ for all $n \geq 0$.

Proof of Claim: Let $h : \mathbb{R} \to \mathbb{R}$ by the function defined by

$$
h(x) = \begin{cases} a_n & , x = n \in \mathbb{Z} \\ (x - n)a_{n+1} + (1 - (x - n))a_n & , n < x < n + 1 \end{cases},
$$

then h is a continuous polygonal function with vertices on $\mathbb{Z}$. Take

$$
d_- = \sup\{d \in (-\infty, -n_0\delta_-] \ : \ h(x) > x\delta_- + d, \ \forall \, x \in [n_0, 0]\}
$$

and

$$
d_+ = \sup\{d \in (-\infty, 0] \ : \ h(x) > x\delta_+ + d, \ \forall \, x \in [0, n_1]\}.
$$

It follows that the graph $(h|_{[n_0, 0]})$ is tangent to the line $L_- = \{y = x\delta_- + d_-\}$ and graph $(h|_{[0, n_1]})$ is tangent to the line $L_+ = \{y = x\delta_+ + d_+\}$. Since the graph of h is a polygonal, the set of tangency between the graph of h and the lines L_- and L_+, that is,

$$
T = \{x \leq 0 \ : \ (x, h(x)) \in L_-\} \cup \{x \geq 0 \ : \ (x, h(x)) \in L_+\}
$$

satisfies $\partial T \subset \mathbb{Z}$.

On one hand, we assume that $d_- \leq d_+$. Hence if we take $N \in [n_0, 0]$ the largest integer in the tangency T, then it is easy to see that

$$h(x) \geq \begin{cases} (x - n_0)\delta_- \geq x\delta_- + d_- & , x \leq n_0 \\ x\delta_- + d_- & , n_0 \leq x \leq N \\ x\delta_- + d_- = (N + n)\delta_- + d_- \geq n\delta_+ + N\delta_- + d_- & , N \leq x = N + n \leq 0, n \geq 0 \\ x\delta_+ + d_+ \geq x\delta_+ + d_- & , 0 \leq x \leq n_1 \\ (x - n_1)\delta_+ \geq x\delta_+ \geq x\delta_+ + d_- & , n_1 \leq x \end{cases} .$$

From the choice of $N \leq 0$ we have that $d_- = h(N) - N\delta_-$, hence

$$h(n + N) \geq \begin{cases} (n + N)\delta_- + d_- = n\delta_- + h(N) & , n \leq 0 \\ (n + N)\delta_+ + d_- \geq n\delta_+ + h(N) & , n \geq 0 \end{cases} . \qquad (19)$$

On the other hand, when $d_+ \leq d_-$, we take $N \in [0, n_1]$ the lowest integer in the tangency T. Also we consider the line $L_0 = \{y = x\delta_- + h(N) - N\delta_-\}$ and $\delta_0 = L_0(0) = h(N) - N\delta_-$. Since $\delta_+ \leq \delta_- < 0$, we have that

$$\delta_0 \leq L_+(0) = d_+ \leq d_- \leq -n_0\delta_- < 0,$$

it follows that

$$h(x) \geq \begin{cases} (x - n_0)\delta_- \geq x\delta_- + \delta_0 & , x \leq n_0 \\ x\delta_- + d_- \geq x\delta_- + \delta_0 & , n_0 \leq x \leq 0 \\ x\delta_+ + d_+ \geq x\delta_- + \delta_0 & , 0 \leq x \leq N \\ x\delta_+ + d_+ & , N \leq x \leq n_1 \\ (x - n_1)\delta_+ \geq x\delta_+ \geq x\delta_+ + d_+ & , n_1 \leq x \end{cases} .$$

Since $d_+ = h(N) - N\delta_+$ and from the choice of δ_0 it is easy to see that we have the same inequality has in inequality (19). Since $a_{n+N} - a_N = h(n + N) - h(N)$ the lemma follows. $\qquad \square$

We define $a = \min\{a_{n_0}, a_{n_1}\}$. Then we have that

a') $a_n \geq (n - n_0)\delta_- + a_{n_0} \geq (n - n_0)\delta_- + a$, for all $n \leq n_0$,

b') $a_n \geq (n - n_1)\delta_+ + a_{n_1} \geq (n - n_1)\delta_+ + a$, for all $n \geq n_1$.

Therefore the sequence $(b_n)_{\mathbb{Z}}$ where $b_n = a_n - a$ satisfies hypothesis of the Claim. Since $a_{n+N} - a_N = b_{n+N} - b_N$, we prove the corollary. $\qquad \square$

Proof of Theorem 6.1. From Corollary 5.1 taking $a_n = g(n, \xi_x)$ and $\delta_\pm = \log(\beta_\pm)$ there exists $|n| \leq N$ such that

$$a_{\pm k + n} - a_n \geq \pm k\delta_\pm$$

for all $k \geq 0$. Then

$$\beta_\pm^{\pm k} \leq g(k + n, M^n\xi_x) \cdot g(n, \xi)^{-1} = g(k, M^n\xi_x) \cdot g(n, \xi) \cdot g(n, \xi)^{-1} = g(k, M^n\xi_x),$$

that is, $f^n(x)$ is a β–critical point. $\qquad \square$

6.1 Critical Points versus Block of Domination

In the seminal work of Pujals and Rodriguez Hertz (vide [18]), critical point are defined as a point such that $x \in H^-(\beta)$ and $f^n(x) \notin \overset{\circ}{H}{}^-(\beta)$ for every $n \geq 1$, where $0 < \beta < 1$. This definition is coherent with the characterization given by Lemma 5.4: *Let $K \subset X$ be a f–invariant set. If $K \subset \overset{\circ}{H}{}^-(\beta)$, then K has dominated splitting.* Moreover, if every point of K have an infinity (for the future) of iterates in $\overset{\circ}{H}{}^-(\beta)$, then X have dominated splitting. Then it is necessary, to think in an obstruction for domination, that the positive orbit of a point not is contained in $\overset{\circ}{H}{}^-(\beta)$.

In this subsection, we relate the notions of β–critical point at the time (n_-, n_+) and β–post–critical point, with the block of domination.

Proposition 6.1. *For $i = 1, 2$, let $\beta_i > 0$. Assume that $x \in H^-(\beta_1)$ with critical direction ξ_x and let $l > 0$. Then the following statements occur:*

   i) If $f^l(x) \in H^-(\beta_2)$ (or in $\overset{\circ}{H}{}^-(\beta_2)$), then $\xi_{f^l(x)} = M^l(\xi_x)$, that is, $M^l(\xi_x)$ is the critical direction for $f^l(x)$.

   ii) If $g(l, \xi_x) > \beta_2^{-l}$ then $f^l(x) \notin \overset{\circ}{H}{}^-(\beta_2)$.

   iii) If $g(l, \xi_x) \geq \beta_2^l$ then $f^l(x) \notin \overset{\circ}{H}{}^-(\beta_2)$.

Proof. Let v_x and $v_{f^l(x)}$ be unitary vectors such that $[v_x] = \xi_x$, $[v_{f^l(x)}] = \xi_{f^l(x)}$ and $\measuredangle(v_{f^l(x)}, A_x^l v_x) \leq \pi/2$ is minimal. To prove *(i)*, we claim that $\measuredangle(v_{f^l(x)}, A_x^l v_x) = 0$. In fact, if $\measuredangle(v_{f^l(x)}, A_x^l v_x) > 0$ then for every $n < -l$ we have that

$$g(n, M^l(\xi_x)) = g(n + l, \xi_x) g(-l, M^l(\xi_x)) \geq \beta_1^{n+l} g(-l, M^l(\xi_x)).$$

Defining $\beta = \max(\beta_1, \beta_2)$ we have that for any $n \leq 0$

$$g(-l, M^l(\xi_x))\beta^{2n+l} \leq g(-l, M^l(\xi_x))\beta_1^{n+l}\beta_2^n \leq g(n, M^l(\xi_x))g(n, \xi_{f^l(x)})$$
$$= \left(\frac{\sin(\measuredangle(A_{f^l(x)}^n v_{f^l(x)}, A_{f^l(x)}^{n+l} v_x))}{\sin(\measuredangle(v_{f^l(x)}, A_x^l v_x))} \right)^2$$
$$< \frac{1}{\sin(\measuredangle(v_{f^l(x)}, A_x^l v_x))},$$

that is a contradiction.

To prove assertion *(ii)*, suppose that $f^l(x) \in \overset{\circ}{H}{}^-(\beta_2)$. Hence it follows from *(i)* that for every $n \leq 0$, $g(n, M^l(\xi_x)) > \beta_2^n$. In particular $g(-l, M^l(\xi_x)) > \beta_2^{-l}$, hence

$$\beta_2^l < g(-l, M^l(\xi_x)) = \frac{1}{g(l, \xi_x)} < \beta_2^l$$

that is a contradiction.

To prove assertion *(iii)*, suppose that $f^l(x) \in \mathring{H}^-(\beta_2)$. Then arguing as in the previous assertion, we have that

$$\beta_2^{-l} < g(-l, M^l(\xi_x)) = \frac{1}{g(l, \xi_x)} \leq \frac{1}{\beta_2^l} = \beta_2^{-l}$$

also a contradiction. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Critical points at the times (n_-, n_+) can be related with a version in terms of block of dominations.

Theorem 6.2. *If $x \in X$ is a β–critical point at the times (n_-, n_+), then*

a) $f^{n_-}(x) \in H^-(\beta_-)$.

b) $f^{n+n_+}(x) \notin \mathring{H}^-(\beta_+)$ *for each $n \geq 1$.*

On the other hand, if $\beta = (\beta_0, \beta_0)$ and x satisfies the item (a) and (b), then x is a β–post–critical point of order $n_+ - n_-$.

Proof. If x is a β–critical point at the times (n_-, n_+), then there exist a direction ξ_x such that for every $n \geq 0$ we have that $g(\pm n, M^{n\pm}\xi_x) \geq \beta_\pm^{\pm n}$. Since

$$g(-n, M^{n_-}\xi_x) \geq \beta_-^{-n}$$

for all $n \geq 0$ it follows that $f^{n_-}(x) \in H^-(\beta_-)$.

On the other hand, taking

$$\beta_0 = \max\{\beta_-, \sqrt[k]{g(k, M^{n_+-k}\xi_x)} : k = 1, \ldots, n_+ - n_-\}$$

we have that $f^{n_+}(x) \in H^-(\beta_0)$. Since $g(n, M^{n_+}\xi_x) \geq \beta_+^n$ for each $n \geq 0$, it follows from Proposition 6.1 that $f^{n+n_+}(x) \notin \mathring{H}^-(\beta_+)$.

Now suppose that x satisfy the items *(a)* and *(b)* with $\beta = (\beta_0, \beta_0)$ and denote the critical direction of x by ξ_x. Without loss of generality we can assume that $n_- = 0$. Let $0 \leq l \leq n_+$ maximal with the property $f^l(x) \in H^-(\beta_0)$. We claim that, for every $n \geq 1$,

$$g(-n, M^{l+n}\xi_x) < \beta_0^{-n}.$$

Assuming the preceding is true, we have that $g(n, M^l\xi_x) \geq \beta_0^n$ for every $n \geq 0$ which prove the Theorem. It remains to prove our claim.

First, we suppose that $g(-1, M^{l+1}\xi_x) \geq \beta_0$. Then for every $k \geq 1$ we have that

$$g(-k, M^{l+1}\xi_x) = g(-(k-1), M^l\xi_x) \cdot g(-1, M^{l+1}\xi_x) \geq \beta_0^{-(k-1)}\beta_0^{-1} = \beta_0^{-k}$$

that implies that $f^{l+1}(x) \in H^-(\beta_0)$, and this contradicts the maximality of l.

Now, we assume that our claim is true for each $0 < n < m$.

Finally, we suppose that $g(-m, M^{l+m}\xi_x) \geq \beta_0^{-m}$. Then for each $0 < k \leq m$ we have that

$$\beta_0^{-m} \leq g(-m, M^{l+m}\xi_x) = g(-k, M^{l+m}\xi_x) \cdot g(-(m-k), M^{l+m-k}\xi_x)$$
$$< g(-k, M^{l+m}\xi_x)\beta_0^{m-k}$$

hence $g(-k, M^{l+m}\xi_x) \geq \beta_0^{-k}$. Similarly, if $k > m$, we have that

$$g(-k, M^{l+m}\xi_x) = g(-m, M^{l+m}\xi_x) \cdot g(-(k-m), M^l\xi_x) > \beta_0^{-m}\beta_0^{-(k-m)} = \beta_0^{-k},$$

that is, for all $k \geq 0$ we have that $g(-k, M^{l+m}\xi_x) \geq \beta_0^{-k}$ that is a contradiction with the maximality of l.

$\square$

Remark 9. *From the preceding Theorem, we have that x is a (β_0, β_0)–critical point, if and only if $x \in H^-(\beta_0)$ and $f^n(x) \notin \mathring{H}^-(\beta_0)$, that is the original definition in [18].*

6.2   The Critical Set

In this section we explain the main properties of $\mathrm{Crit}(\beta)$. These properties justify the notion of critical point, show how the notion of critical point is an intrinsic notion of the dynamics, and highlight its meaning.

• **Compactness:** *We recall that the set of critical point is a compact set.*

In fact, let $(x_k)_{\mathbb{N}} \subset \mathrm{Crit}(\beta)$ such that $x_k \to x$, and denote his critical directions by ξ_k. By passing to subsequence if necessary, there exist a direction $\xi \in \overline{\mathbb{C}}_x$ such that $\xi_k \to \xi$. Since for each $n \geq 0$ we have that $g(\pm n, \xi_k) \geq \beta_{\pm}^{\pm n}$, taking k goes to infinity we conclude that $x \in \mathrm{Crit}(\beta)$.

• **Distinguished Critical Point:** *We assert that, if $\beta \in \Delta$ with $\beta_+ > b$, then in the orbit of a critical point x, there exist a critical point positively maximal.*

In others words, if $x \in \mathrm{Crit}(\beta)$ there exist $n_0 \geq 0$ such that $f^n(f^{n_0}(x)) \notin \mathrm{Crit}(\beta)$ for each $n \geq 1$ (We call the maximal element in the orbit of a critical point, by **distinguished critical point**).

In fact, suppose by contradiction that there exists $(n_k) \nearrow \infty$ such that $f^{n_k}(x) \in \mathrm{Crit}(\beta)$. Without loss of generality, we can take $n_k \to \infty$ such that the limit

$$\mu = \lim_{k\to\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \delta_{M^i(\xi_x)}$$

there exists. Denote the projection in the first variable of μ by μ'.

On the other hand, since $g(-n_k, M^{n_k}\xi) \geq \beta_-^{-n_k}$ then the inequality

$$\beta_+^{n_k} \leq g(n_k, \xi) \leq \beta_-^{n_k} < 1 < \beta_-^{-n_k} \tag{20}$$

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holds.

We recall that

$$\operatorname{supp}(\mu) = \bigcap_{k \geq 1} \overline{\{(f^{n_s}(x), M^{n_s}\xi_x) \ : \ s \geq k\}}.$$

Hence, arguing as in the proof of Criteria of Negative Exponent, and since the equation (20) holds, we can conclude that for every (z, w) with $x \in \mathcal{R}(A, \mu')$ the limit $I(z, w)$ defined in the equation (15) satisfy

$$\log(\beta_+) \leq I(z, w) \leq -\log(\beta_-).$$

Since $\beta_+, \beta_- > b$ and f has no attractor, then working in the same way as in Criteria of Negative Exponent we obtain a contradiction.

• **Change of Metric:** Let $(\cdot|\cdot)_i$ be a hermitian metric in TX, where $i = 0, 1$. Denotes his spherical metrics related with them by $|| \cdot ||_i$ (see Appendix for details), and denote the set of critical points by $\operatorname{Crit}_i(\beta)$.

We claim that there exist a positive integer N such that

$$\operatorname{Crit}_0(\beta) \subset \bigcup_{j=-N}^{N} f^j(\operatorname{Crit}_1(\beta)).$$

In other words, every β–critical point to g_0, is a β–post–critical point for g_1 of order N.

In fact, let $\alpha > 0$ such that $\alpha^{-1}|| \cdot ||_1 \leq || \cdot ||_0 \leq \alpha|| \cdot ||_1$. We recall from Definition 3, equation (16), and Definition 4, that for $\xi \in \overline{\mathbb{C}}_x$, $n \in \mathbb{Z}$ and for every $w \in T_\xi \overline{\mathbb{C}}_x$ we have that

$$g_i(n, \xi) = ||(M^n)'(\xi)||_i = \frac{||(M^n)'(\xi)w||_{i, R^n(\xi)}}{||w||_{i, \xi}}.$$

Replacing the previous equation we have

$$\frac{g_1(n, \xi)}{g_0(n, \xi)} = \frac{||(M^n)'(\xi)w||_{1, R^n(\xi)}}{||(M^n)'(\xi)w||_{0, R^n(\xi)}} \cdot \frac{||w||_{0, \xi}}{||w||_{1, \xi}}$$

and hence we conclude that $\alpha^{-2} \leq g_1/g_0 \leq \alpha^2$. It is easy to see that if $\alpha \leq 1$ then every β–critical point to g_0, is a β–critical point for g_1 and reciprocally. Then we may assume that $\alpha > 1$.

Let $x \in \operatorname{Crit}_0(\beta)$. Let $\alpha_0 = 2\log(\alpha)$, let $\delta_\pm = \log(\beta_\pm)$ and let $a_n = \log(g_1(n, \xi))$. If we take $L_\pm(x, d) = x\delta \pm +d$ then for each $n \geq 0$ we have that

$$\log(g_0(\pm n, \xi)) + \alpha_0 \geq \log(g_1(\pm n, \xi)) \geq \log(g_0(\pm n, \xi)) - \alpha_0 \geq L_\pm(\pm n, -\alpha_0).$$

Take $\alpha_1 = -\alpha_0\delta_+/\delta_-$. Denote the function floor and ceiling by $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ respectively. Since the lines $L_-(x, -\alpha_0)$ and $L_+(x, \alpha_1)$ is the point $\{(\alpha_0/\delta_-, 0)\}$, then there exist

$$d_+ = \sup\{d \in [\alpha_1, \alpha_0] : a_n \geq n\delta_+ + d, \ \forall \ n \geq \lceil -d/\delta_+\rceil\}$$

and

$$d_- = \sup\{d \in [-\alpha_0, -d_+/\delta_+] : a_n \geq n\delta_- + d, \ \forall \ n \leq \lfloor -d/\delta_-\rfloor\}.$$

Define $n_+ = \lceil -d_+/\delta_+\rceil$ and $n_- = \lfloor -d_-/\delta_-\rfloor$. From the choice of the constants, it is not difficult to see that

$$a_n \geq (n - n_+)\delta_+, \ \text{where } n \geq n_+ \ \text{ and } \ a_n \geq (n - n_-)\delta_-, \ \text{where } n \leq n_-.$$

It follows that we are in the hypothesis of the Claim in the proof of Lemma 5.1, then there exists $N' \in [n_-, n_+]$ such that $a_{\pm n + N'} - a_{N'} \geq \pm\delta_\pm$ for all $n \geq 0$. This we conclude that $f^{N'}(x) \in \mathrm{Crit}_1(\beta)$. Finally, since

$$\left\lfloor \frac{\alpha_0}{\delta_-} \right\rfloor \leq n_- \leq n_+ \leq \left\lceil -\frac{\alpha_0}{\delta_+} \right\rceil$$

we conclude that taking $N = \max\left\{ \left|\left\lfloor\frac{\alpha_0}{\delta_-}\right\rfloor\right|, \left|\left\lceil-\frac{\alpha_0}{\delta_+}\right\rceil\right| \right\}$, we have our assertion.

• **Conjugated Cocycles:** As the notion of dominated splitting is invariant by conjugation, we can expect a similar property to the notion of critical point. More precisely, we state:

Lemma 6.1. *Let A and B be two conjugated lineal cocycles over TX. Then β–critical points of A becomes β–post–critical points of B.*

Proof. In fact, let $A = (f_0, A_*)$, $B = (f_1, B_*)$ and $H = (h, H_*)$ be linear cocycles and M, N and L, the respective projective cocycles related with them. Assume that $H \circ A = B \circ H$. Denote the norm of the multiplier related with A (resp. B) by g_0 (resp. g_1).
Then we have that

$$g_0(n, \xi) = ||(M^n)'(\xi)|| \leq ||(L^{-1})'(N^n(L(\xi)))|| \cdot ||L'(\xi)|| \cdot ||(N^n)'(L(\xi))||$$
$$= ||(L^{-1})'(N^n(L(\xi)))|| \cdot ||L'(\xi)|| \cdot g_1(n, L(\xi)).$$

Hence taking $c_\pm = \sup\{||(L^{\pm 1})'(\varpi)|| : \varpi \in \mathbb{P}(X)\}$ and $c = c_- \cdot c_+$ we conclude that $g_0(n, \xi) \leq cg_1(n, L(\xi))$. Of similar way, we can conclude that $g_1(n, \xi) \leq cg_0(n, L^{-1}(\xi))$ and therefore $c^{-1} \leq g_1(n, L(\xi))/g_0(n, \xi) \leq c$.
Arguing as in the preceding item, we can conclude that if x is a critical point for A, then $h(x)$ is a post–critical point of order bounded. $\square$

6.3 Proof of Theorem 4.1

Proof. Let $1 > \beta > b$ and $1 \leq \gamma < b^{-1}\beta$. We can choice α such that $b < \alpha < \gamma^{-1}\beta$. Let ν be a f–invariant measure and let $x \in \mathcal{R}(A, \nu)$ (see, Subsection 2.2), then

$$\lim_{n \to +\infty} \frac{1}{n} \log \big(g(n, E_x)\big) = \lambda^+(x) - \lambda^-(x) \geq -\lambda^-(x) \geq -\log(b).$$

We consider

$$1 > \alpha > c > b$$

arbitrarily but fixes. It follows that for m large enough we have that

$$g(m, E_x) \geq c^{-m}.$$

From Pliss's Lemma, there exists a sequence $(m_k)_k \nearrow \infty$ satisfying

$$g(n, M^{m_k}(E_x)) \geq \alpha^{-n} > \gamma^n \beta^{-n} > \gamma \beta^{-n} \text{ for every } n \geq 1.$$

It follows that $\gamma H^+(\beta)$ is a not empty set. Moreover, $\gamma H^+(\beta)$ contains all accumulation points of the set $(f^{m_k}(x))_k$, which critical directions is an accumulation point of the set $(M^{m_k}(E_x))_k$.

Arguing in the same way, for $x \in \mathcal{R}(A, \nu)$ we have that

$$\lim_{n \to +\infty} \frac{1}{n} \log \big(g(-n, F_x)\big) = \lambda^+(x) - \lambda^-(x) \geq \lambda^-(x) \geq -\log(b),$$

and we can find a sequence $(n_k)_k \nearrow \infty$ such that

$$g(-n, M^{-n_k}(F_x)) \geq \beta^{-n} > \gamma^{-1}\beta^{-n} \text{ for every } n \geq 1,$$

and conclude that $H^-(\beta)$ is not empty.

To see the compactness of $\gamma^{-1}H^+(\beta)$, we take a sequence $(y_n)_n \subset H^+(\beta)$ which critical direction $(\varpi_n)_n$. If y is any accumulation point of $(y_n)_n$, then (taking subsequence if necessary) there exists a direction ϖ_y accumulated by the directions $(\varpi_n)_n$ that satisfy $g(n, \varpi_x) \geq \gamma \beta^{-n}$, then $y \in \gamma H^+(\beta)$.

Finally, let $X_0^+ = \cup_{n \in \mathbb{Z}} f^n(\gamma H^+(\beta))$. Note that for any regular point $x \in \mathcal{R}(A, \nu)$, there exists a forward iterate of x in $\gamma H^+(\beta)$, then $\mathcal{R}(A, \nu) \subset X_0^+$ and hence, X_0^+ have total measure. $\square$

6.4 Dynamically Defined Cocycles

In this subsection, we let f be a biholomorphisms in a two dimensional complex manifolds (for example, a generalized Hénon map), with a set X compact and f–invariant. We also consider the natural cocycle related with f, that is $Df_\# = (f, Df)$ the cocycle defined on TX by the function and his derivative. We also assume that f is b–asymptotically dissipative and has no attractor.

We recall that in the one dimensional context, both real and complex, critical point are far from hyperbolic set, however they can accumulated by hyperbolic sets.

In the two dimensional context, a similar result holds:

Corollary 6.1. *Suppose that X does not have dominated splitting, but there exist an f–invariant compact set $X' \subset X$ such that X' is hyperbolic and/or has dominated splitting. Then for every β with $\beta_+ > b$, $\mathrm{dist}(X', \mathrm{Crit}(\beta)) > 0$.*

In fact, this follows from the Main Theorem, since both X' and $\mathrm{Crit}(\beta)$ are compact and disjoints. Moreover, we also can state:

Lemma 6.2. *A critical point is not a regular point.*

Proof. Let x be a β–critical point with critical direction ξ and $\beta_- \geq \beta_+ > b$. We assume that x is a regular. Let $T_x = E^+ \oplus E^-$ a splitting related with the Lyapunov exponents. We also have the inequalities

$$\lambda^- \leq \log(b) < 0 \leq \lambda^+,$$

and

$$\lambda^- - \lambda^+ \leq \log(b).$$

We assert that the direction related with ξ is the subspace E^+. In fact, if not, we have

$$\lim_{n \to -\infty} \frac{1}{n} \log \|Df_z^n v_\xi\| = -\lambda^-,$$

where v_ξ is unitary and define the direction ξ. From the previous equation and since that $g(-n, \xi) \geq \beta_-^{-n}$ for $n \geq 0$, we have that

$$\lim_{n \to -\infty} \frac{1}{n} \log(g(n, \xi)) = \lambda^- - \lambda^+ \geq -\log(\beta_-),$$

and this implies that $b \geq \beta_-^{-1}$ that is a contradiction. Now, as E^+ define the direction ξ, we have that

$$\lim_{n \to \infty} \frac{1}{n} \log(g(n, \xi)) = \lambda^- - \lambda^+.$$

Since $g(n, \xi) \geq \beta_+^n$ we conclude hat

$$\log(b) \geq \log(\beta_+),$$

that is a contradiction. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Moreover, critical point are disjoint to *"hyperbolic blocks"*. For example, let f be a Hénon map. We denote by $\mathcal{R} \subset J^*$ the set of all regular points. Given $C > 0$ fixed, consider the set

$$B(0, C) = \big\{z \in \mathcal{R} : |Df^n|E^-(z)| \leq C\exp(n\lambda^-(z))$$
$$\text{and } |Df^{-n}|E^+(z)| \leq C\exp(-n\lambda^+(z))\big\}.$$

Clearly, this set is closed, and given $l \in \mathbb{N}$ we define the *hyperbolic block* of large l

$$B(l, C) = \cup_{k=-l}^{l} f^k(B(0, C)).$$

Since all point in the hyperbolic block is regular, it is follows from the previous lemma, that a critical point is disjoint to the blocks.

• **Tangencies of a Periodic Point Contain a Critical Point:** Let f be a Hénon map with $b = |\det(Df)| < 1$. Let p be a periodic point of f and q be a point of tangencie between the stable and the unstable manifolds of p. *We assert that in the orbit of $\mathcal{O}(p)$ there exist a β–critical point, when $\beta \in \Delta$ and $b_+ > b$.*

In fact, without loss of generality, we can assume that p is a fixed point. We denote the local stable/unstable manifold of p of size ε by $W^s_\varepsilon(p)$ and $W^u_\varepsilon(p)$ respectively. Let λ^s and λ^u the eigenvalues of Df in p, then $b = |\lambda^s| \cdot |\lambda^u|$.

Note that for each $n \geq 0$

$$g(-n, E^u_p) = \frac{b^{-n}}{|Df^{-n}|E^u_p|^2} = \left(\frac{|\lambda^u|^2}{b}\right)^n > b^{-n} > \beta^{-n}_-,$$

and that

$$g(n, E^s_p) = \frac{b^n}{|Df^n|E^s_p|^2} = \frac{|\lambda^u|^n}{|\lambda^s|^n} = \left(\frac{|\lambda^u|^2}{b}\right)^n > b^{-n} > 1 > \beta^n_+.$$

We can take $\varepsilon > 0$ is small enough such that for every $z \in W^s_p(\varepsilon)$ (resp. $z \in W^u_p(\varepsilon)$) we have that $z \approx p$ and $T_z W^s_p(\varepsilon) \approx E^s_p$ (resp. $T_z W^u_p(\varepsilon) \approx E^u_p$). We can conclude that for each $z \in W^u_p(\varepsilon)$ (resp. $z \in W^s_p(\varepsilon)$) we have that $g(-n, T_z W^u_p(\varepsilon)) \geq \beta^{-n}_-$ (resp. $g(n, T_z W^s_p(\varepsilon)) \geq \beta^n_+$) for each $n \geq 0$.

Finally, let q_u be the first iterate to the past of q that is inside of $W^u_p(\varepsilon)$ and let $n_+ > 0$ such that $f^{n_+}(q_u)$ is the first iterate to the future of q that is inside of $W^s_p(\varepsilon)$. Since, q is a tangencie point we have that $Df^{n_+}(T_{q_u} W^u_p(\varepsilon)) = T_{f^{n_+}(q_u)} W^s_p(\varepsilon)$, hence we conclude that q_u is a β–critical point at the times $(0, n_+)$. From Theorem 6.1 we conclude that there exist $0 \leq n_0 \leq n_+$ such that $f^{n_0}(q_u)$ is a β–critical point, so we have our assertion.

6.5   Some Remark on Critical points for Hénon maps

Recall that a polynomial or a rational maps in $\mathbb{C}$, always have critical points. Moreover, the critical points determine the global dynamics of a polynomial. Whit this in mind, we can state: *Let p by a polynomial over $\mathbb{C}$ of degree at least two. Then the Julia set $J_p \subset \mathbb{C}$ is hyperbolic, if and only if, $PC(p) \cap J_p = \emptyset$.* Here $PC(p)$ denote the post–critical set defined by

$$PC(p) = \overline{\cup_{n \geq 1} p^n \left(\{z : p'(z) = 0\}\right)}.$$

Following these result, we wonder if for a complex Hénon maps, always there exists critical point (even outside of the Julia). Moreover, if these "critical points" there exist, we wonder if they determine the global dynamics. In fact, recall that we have proved: *The intersection $Crit \cap J = \emptyset$, if and only if J has dominated splitting, where $Crit$ denote the critical set.*

If we think in the words "critical point" as the object that represent the obstruction to have dominated splitting (independent of the adopted definition), we can formulate:

Question A: *Do always exists critical point in $\mathbb{C}^2$?*

Question B: *If K^+ has interior, always exists critical point in K^+?*

We can answer positively the Question B, for a polynomial automorphisms close to the one dimensional polynomial p. Let

$$f_\delta(x, y) = (y, p(y) - \delta x),$$

with $|\delta|$ small. When we refer to critical point of p, i.e., $p'(x) = 0$ we denote them as one dimensional critical point.

Let us assume that the polynomial p satisfies:

1. there are not one dimensional critical point in J_p,

2. J_p is connected,

3. the filled Julia set K_p has interior.

The item 3, implies that the set K_p^+ associated with the two dimensional map $f_0 : (x, y) \mapsto (y, p(y))$, has non empty interior. In fact, is easy to see that $K_p^+ = \mathbb{C} \times K_p$. We recall that since $|\delta|$ small, $f = f_\delta$ is close to f_0, hence J_f is close to the set $J_0 = \{(y, p(y)) : , y \in J_p\}$.

Proposition 6.2. *Under the previous hypothesis, there are a critical point in the interior of K^+.*

Proof. If there are not critical points in K^+, then this has dominated splitting, so K^+ is foliated by holomorphic stable leaves

$$K^+ = \sup_{x \in K^+} W^s(x).$$

On the other hand, the map

$$z = (x, y) \hookrightarrow (y, p(y)) \hookrightarrow p(y),$$

is holomorphic and the image of K_p^+ is K_p, that is contained in the y-axis. So, for $|\delta|$ small enough, there exists a holomorphic disc D, close to the y-axis and transversal to the stable foliation of f in K^+.

We define π^s the projection to D, by the stable foliation. Now we define

$$z \in D \cap K^+ \hookrightarrow f(z) \hookrightarrow \pi^s(f(z)) \in D.$$

Then the map $(\pi^s \circ f) : D \to D$ is a holomorphic one dimensional map. We denote by pr_2 the projection in the second variable. Since π^s close to pr_2 nearby

the Julia set, and f close to f_0, then $\pi^s \circ f$ is close to p nearby the Julia set, thus it is follows that $\pi^s \circ f$ has degree equal to degree of polynomial p.

From the previous observation, there exist $c \in D$ such that $(\pi^s \circ f)'(c) = 0$. Now, Df does not have kernel, it is follows that

$$(Df)(T_cD) \subset \text{Kernel}(D\pi^s).$$

Since that $\text{Kernel}(D\pi^s) = E^s$, we conclude that $(Df)(T_cD) \subset E^s$ but this is a contradiction, because $T_cD \pitchfork E^s$. $\qquad\square$

Therefore, one of the delicate step, is to give a formal definition of critical point in K^+ in a general context. All of this, in the direction of a possible state of the following:

Conjecture: *If the Julia set there are not β–critical point, and if the "critical points" of the interior of K^+ not accumulate on J (thinking as in the one–dimensional post–critical point), then J is hyperbolic.*

A   Appendix

A.1   Hermitian and Spherical metrics

This section is devoted to proving the existence of a spherical metric in the Riemann sphere, given an hermitian metric. For this purpose, it is suffices to make this construction in $\mathbb{C}^2$.

The Riemann sphere is the projective space consisting of all 1–dimensional subspaces of $\mathbb{C}^2$ or complex lines. The complex line that through point v is the set $[v] = \{\lambda v : \lambda \in \mathbb{C}^*\}$. Writing the point $[v]$ in homogeneous coordinates this has the form $[z_1 : z_2]$ where $v = (v_1, v_2)$. So we obtain that

$$\overline{\mathbb{C}} = \big\{[z_1 : z_2] : (z_1, z_2) \in \mathbb{C}^2\big\}.$$

Each $z \in \mathbb{C}$ is related with $[z_1 : z_2]$ if and only if $[z_1 : z_2] = [z_1/z_2 : 1] = [z : 1]$. The point at infinity is related with the class $[1 : 0]$, and we can write $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. In this coordinates the **standard spherical metric**

$$d\rho = \frac{2|dz|}{1 + |z|^2} \tag{21}$$

and has constant Gaussian curvature $+1$.

The previous construction was made under the representation in homogenous coordinates in the canonical base. Now, we will repeat this construction, but considering an arbitrary base, and we will find the relationship between this different representations.

Let $\beta = \{v_1, v_2\}$ be a base of $\mathbb{C}^2$ and write $v = w_1 v_1 + w_2 v_2 = (w_1, w_2)_\beta$. We write the homogeneous coordinate in the base β of the vector v as $[w_1 : w_2]_\beta$. Also we relate each $[w_1 : w_2]_\beta$ with the point $w \in \mathbb{C}$ if and only if $w = w_1/w_2$. Finally, we denote

$$\overline{\mathbb{C}}_\beta = \big\{[w_1 : w_2]_\beta : w_1 v_1 + w_2 v_2 \in \mathbb{C}^2\big\},$$

and define the **spherical metric in the base β on $\overline{\mathbb{C}}_\beta$** by the equation

$$d\rho_\beta = \frac{2|dw|}{1 + |w|^2}.$$

On the other hand, let L the linear transformation satisfying $Lv_i = e_i$ whit $i = 1, 2$ and where $\{e_1, e_2\}$ denote the canonical base. It is not difficult to see that, denoting the Möbius transformation related with L by N, then we have that $N(z) = w$.

Let $v_z = (z_1, z_2)$ such that $z = z_1/z_2$. From equation (21) it not difficult to see that

$$d\rho = 2|dz|\frac{|(v_z|e_2)|^2}{(v_z|v_z)}$$

where $(\cdot|\cdot)$ denote the standard hermitian metric. Similarly, if $(\cdot|\cdot)_0$ is a hermitian metric in $\mathbb{C}^2$ such that β is a orthonormal bases, then

$$d\rho_\beta = 2|dw|\frac{|(v_w|v_2)_0|^2}{(v_w|v_w)_0} \tag{22}$$

where $v_w = (w_1, w_2)_\beta$ such that $w = w_1/w_2$.

To justify that the definition given in equation (22) is a good definition, it is necessary to prove that:

> (†) **If β is a orthonormal base (different of the canonical base)**
> **in the standard hermitian metric,**
> **then $d\rho = d\rho_\beta$.**

In fact, if β is a orthonormal base standard hermitian metric, then L is an isometry in the hermitian metric and the induced Möbius transformation N is an isometry in the standard spherical metric. Write

$$L = \begin{pmatrix} \overline{b} & -a \\ \overline{a} & b \end{pmatrix}$$

and

$$N(z) = \frac{\overline{b}z - a}{\overline{a}z + b}.$$

Let $v_z = (z_1, z_2)$ with $z = z_1/z_2$. Since $N(z) = w$, then we can take $v_w = (w_1, w_2)_\beta = w_1v_1 + w_2v_2 = (\overline{b}z - a, \overline{a}z + b)_\beta$.

Using the notation above and the equation (22) we conclude that

$$d\rho = 2|dz|\frac{|(v_z|e_2)|^2}{(v_z|v_z)} = 2|dz|\frac{|z_2|^2}{|z_1|^2 + |z_2|^2}$$

$$= 2|dz|\frac{|z_2|^2}{(Lv_z|Lv_z)}$$

$$= 2|dz|\frac{|z_2|^2}{|\overline{b}z_1 - az_2|^2 + |\overline{a}z_1 + bz_2|^2}$$

$$= 2|dz|\frac{1}{|\overline{b}z - a|^2 + |\overline{a}z + b|^2}$$

$$= \frac{2|dz|}{1 + \left|\dfrac{\overline{b}z - a}{\overline{a}z + b}\right|^2} \cdot \left|\frac{1}{(\overline{a}z + b)^2}\right|$$

$$= \frac{2|dz|}{1 + |N(z)|^2} \cdot |N'(z)|$$

$$= 2\frac{|dw|}{1 + |w|^2},$$

and note that

$$\frac{|(v_w|v_2)|^2}{(v_w|v_w)} = \frac{|w_2|^2}{|w_1|^2 + |w_2|^2} = \frac{1}{1 + |w_1/w_2|^2} = \frac{1}{1 + |w|^2}.$$

Hence (†) holds.

Finally, equation (22) allows to define **the spherical metric** as an intrinsic object of the hermitian metric (prefixing an orthonormal base, but not depending of this base). With this, we can justify the existence of a spherical metric in a projective bundle in terms of the hermitian metric defined in the fibre bundle.

References

[1] E. Bedford; M. Lyubich; J. Smillie. *Polynomial diffeomorphisms of C^2. IV. The measure of maximal entropy and laminar currents.* Invent. Math. 112 (1993), no. 1, 77–125.

[2] E. Bedford; J. Smillie. *Polynomial diffeomorphisms of C^2: currents, equilibrium measure and hyperbolicity.* Invent. Math. 103 (1991), no. 1, 69–99.

[3] E. Bedford; J. Smillie. *Polynomial diffeomorphisms of C^2. II. Stable manifolds and recurrence.* J. Amer. Math. Soc. 4 (1991), no. 4, 657–679.

[4] E. Bedford; J. Smillie. *Polynomial diffeomorphisms of $\mathbb{C}^2$. III. Ergodicity, exponents and entropy of the equilibrium measure.* Math. Ann. 294 (1992), no. 3, 395–420.

[5] M. Benedicks; L. Carleson. *The dynamics of the Hénon map.* Annals of Math., 133 (1991), 73–169.

[6] J. Conway. *Functions of one complex variable. Second edition.* Graduate Texts in Mathematics, 11. Springer–Verlag, New York–Berlin, 1978. xiii+317 pp.

[7] S. Crovisier. *Sur la notion de Criticalité de Pujals-Rodriguez Hertz.* Preprint.

[8] J. E. Fornæss. *The Julia set of Hénon maps.* Math. Ann. 334 (2006), no. 2, 457–464.

[9] J. E. Fornæss; N. Sibony. *Complex Hénon mappings in C^2 and Fatou-Bieberbach domains.* Duke Math. J. 65 (1992), no. 2, 345–380.

[10] S. Friedland; J. Milnor. *Dynamical properties of plane polynomial automorphisms.* Ergodic Theory Dynam. Systems 9 (1989), no. 1, 67–99.

[11] Hubbard, John H. *The Hénon mapping in the complex domain.* Chaotic dynamics and fractals (Atlanta, Ga., 1985), 101–111, Notes Rep. Math. Sci. Engrg., 2, Academic Press, Orlando, FL, 1986.

[12] D. Husemoller. *Fibre bundles.* Third edition. Graduate Texts in Mathematics, 20. Springer-Verlag, New York, 1994. xx+353 pp.

[13] J. Hubbard; R. Oberste-Vorth. *Hénon mappings in the complex domain. I. The global topology of dynamical space.* Inst. Hautes tudes Sci. Publ. Math. No.79 (1994), 5–46.

[14] Hubbard, John H. ; Oberste-Vorth, Ralph W. *Hénon mappings in the complex domain. II. Projective and inductive limits of polynomials.* Real and complex dynamical systems (Hillerød, 1993), 89–132, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 464, Kluwer Acad. Publ., Dordrecht, 1995.

[15] O. Lehto; K. I. Virtanen. *Quasiconformal mappings in the plane.* Second edition. Translated from the German by K. W. Lucas. Die Grundlehren der mathematischen Wissenschaften, Band 126. Springer-Verlag, New York-Heidelberg, 1973. viii+258 pp.

[16] R. Mañé. *Hyperbolicity, sinks and measure in one-dimensional dynamics.* Comm. Math. Phys. 100 (1985), no. 4, 495–524.

[17] J. Milnor. *Dynamics in one complex variable. Third edition.* Annals of Mathematics Studies, 160. Princeton University Press, Princeton, NJ, 2006. viii+304 pp.

[18] E. R. Pujals; F. Rodriguez Hertz. *Critical points for surface diffeomorphisms.* J. Mod. Dyn. 1 (2007), no. 4, 615–648.

[19] E. R. Pujals; M. Sambarino. *Homoclinic tangencies and hyperbolicity for surface diffeomorphisms.* Annals of Mathematics, vol. 151, 2000, p. 961–1023.

[20] M. Viana. *Multiplicative ergodic theorem of Oseledets (after Ricardo Mañé): A proof of Oseledets' Theorem.* Lecture Notes.

[21] F. Valenzuela–Henriquez. *Equivalent conditions for hyperbolicity on partially hyperbolic holomorphic map.* Indiana Univ. Math. J. Preprint.